Exponential Decay of Eigenfunctions and Accumulation of Eigenvalues on Manifolds with Axial Analytic Asymptotically Cylindrical Ends [1]

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Abstract

In this paper we continue our study of the Laplacian on manifolds with axial analytic asymptotically cylindrical ends initiated in arXiv:1003.2538. By using the complex scaling method and the Phragmén-Lindelöf principle we prove exponential decay of the eigenfunctions corresponding to the non-threshold eigenvalues of the Laplacian on functions. In the case of a manifold with (non-compact) boundary it is either the Dirichlet Laplacian or the Neumann Laplacian. We show that the rate of exponential decay of an eigenfunction is prescribed by the distance from the corresponding eigenvalue to the next threshold. Under our assumptions on the behaviour of the metric at infinity accumulation of isolated and embedded eigenvalues occur. The results on decay of eigenfunctions combined with the compactness argument due to Perry imply that the eigenvalues can accumulate only at thresholds and only from below. The eigenvalues are of finite multiplicity.

Key words: spectral geometry, accumulation of eigenvalues, exponential decay of eigenfunctions, thresholds, asymptotically cylindrical ends, complex scaling, Phragmén-Lindelöf principle, Dirichlet Laplacian, Neumann Laplacian

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1 Introduction

Consider a manifold $(\mathcal{M}, \mathsf{g})$ with an asymptotically cylindrical end. This means that $\mathcal{M}$ is a smooth non-compact manifold of the form $\mathcal{M}_c \cup (\mathbb{R}_+ \times \Omega)$, where $\mathcal{M}_c$ is a compact manifold, and $\mathbb{R}_+ \times \Omega$ is the Cartesian product of the

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positive semi-axis $\mathbb{R}_+$ and a compact manifold Ω, see Fig. 1 and Fig. 2. Furthermore, the metric g asymptotically approaches at infinity the product metric $dx\otimes dx+\mathfrak{h}$ on the semi-cylinder $\mathbb{R}_+\times\Omega$, where $\mathfrak{h}$ is a metric on Ω. Traditionally, one studies the Laplacian on manifolds with asymptotically cylindrical ends under more [1,8,23,24] or less [4,6,25,10] restrictive assumptions on the rate of convergence of the metric g to the product metric $dx \otimes dx + \mathfrak{h}$ at infinity. We do not make this kind of assumptions. Instead, in [12] and in this paper we consider manifolds with axial analytic asymptotically cylindrical ends. On these manifolds the metric g extends by analyticity to a conical neighborhood of the axis $\mathbb{R}_+$ of the semi-cylinder $\mathbb{R}_+ \times \Omega$, and the continuation tends at infinity to the analytic continuation of $dx \otimes dx + \mathfrak{h}$; for precise definitions see Section 2. Due to these properties of g the end $(\mathbb{R}_+ \times \Omega, \mathsf{g}\restriction_{\mathbb{R}_+\times\Omega})$ is said to be axial analytic. On manifolds with axial analytic asymptotically cylindrical ends the metric g converges at infinity to $dx \otimes dx + \mathfrak{h}$ with arbitrarily slow rate.

As is known [2,26], the eigenvalues of the Laplacian on manifolds with cylindrical ends have no finite points of accumulation. In the cylindrical ends we have $\mathsf{g}\restriction_{\mathbb{R}_+\times\Omega}= dx \otimes dx + \mathfrak{h}$, and exponential decay of the non-threshold eigenfunctions can be easily seen by separation of variables. Let us also note that finite points of accumulation do not occur on manifolds with asymptotically cylindrical ends, provided that we admit only exponential convergence of the metric g to $dx \otimes dx + \mathfrak{h}$ at infinity [23]. In this case exponential decay of the non-threshold eigenfunctions is a consequence of the asymptotic theory, see e.g. [17,18,23] and references therein. Once we allow for sufficiently slow convergence of g to $dx \otimes dx + \mathfrak{h}$ at infinity, the situation changes and eigenvalues may accumulate at finite distances. Under an assumption on the rate of convergence of the metric at infinity it is possible to prove the Mourre estimates [6]. In particular, these estimates imply that the eigenvalues can accumulate only at thresholds. Power decay of eigenfunctions and eigenvalue accumulation for the Laplacian on functions, for the Dirichlet Laplacian, and for the Neumann Laplacian were studied in [4], where it is assumed that the metric allows for separation of variables in the ends and satisfies an assumption on the rate of convergence at infinity.

In [12] we developed an approach to the complex scaling on manifolds with axial analytic asymptotically cylindrical ends and established a variant of the Aguilar-Balslev-Combes theorem for the Laplacian Δ on functions. In particular, we proved that the Laplacian has no singular continuous spectrum, all non-threshold eigenvalues are of finite multiplicity, and the eigenvalues of Δ can accumulate only at thresholds. In this paper we continue to study the Laplacian and prove the following: 1) Any non-threshold eigenfunction of Δ decays at infinity with some exponential rate prescribed by the distance from the corresponding eigenvalue to the next threshold of Δ; 2) The eigenvalues of Δ are of finite multiplicity and can accumulate at thresholds only from below.

Besides, we aim to show a certain similarity between methods and results of the theory of N-body Schrödinger operators in $\mathbb{R}^n$ and the analysis on manifolds with axial analytic asymptotically cylindrical ends. It is interesting to note that there is also a connection between the theory of Schrödinger operators and the analysis on symmetric spaces [21,22]. As in [12] we consider three generic cases: 1) Δ is the Laplacian on a manifold without boundary; 2) Δ is the Dirichlet Laplacian on a manifold with noncompact boundary; 3) Δ is the Neumann Laplacian on a manifold with noncompact boundary. We give examples of manifolds with axial analytic asymptotically cylindrical ends demonstrating that the eigenvalues of Δ may indeed accumulate at thresholds.

As is typically the case, it is much easier to prove exponential decay of the eigenfunctions corresponding to the isolated eigenvalues. In fact, this can be done by methods of the asymptotic theory [17,18,20], which work nicely on manifolds with axial analytic asymptotically cylindrical ends (see also [13] and [14, Appendix]). Or, equivalently, one can use the dilation analytic techniques similar to those in the theory of Schrödinger operators e.g. [30, Chapter XII.11]. However, Δ is a non-negative operator, and therefore only the Dirichlet Laplacian may have isolated eigenvalues below its absolutely continuous spectrum $\sigma_{ac}(\Delta) = [\nu, \infty)$, $\nu > 0$. All eigenvalues of the Neumann Laplacian and of the Laplacian on a manifold $(\mathcal{M}, \mathsf{g})$ without boundary are embedded into the spectrum $\sigma_{ac}(\Delta) = [0, \infty)$.

As is well-known, for Schrödinger operators with dilation analytic potentials it is also possible to prove exponential decay of the eigenfunctions corresponding to the non-threshold embedded eigenvalues, see e.g. [30, Chapter XII.11]. It turns out that similar methods, based on the complex scaling and the Phragmén-Lindelöf principle, can be applied on manifolds with axial analytic asymptotically cylindrical ends. This allows us to prove that every eigenfunction corresponding to a non-threshold eigenvalue of the Laplacian is of some exponential decay at infinity. This fact plays a crucial role throughout the paper. Note that due to arbitrarily slow convergence of g to $dx \otimes dx + \mathfrak{h}$ at infinity the asymptotic theory does not give any information on decay of eigenfunctions corresponding to an embedded eigenvalue, see e.g. [13,17,18,20,29]. Nonetheless, one can employ the asymptotic theory in order to refine the rate of exponential decay of eigenfunctions. As a result we conclude that the rate of exponential decay of a non-threshold eigenfunction is prescribed by the distance from the corresponding eigenvalue to the next threshold of the Laplacian. Similar results for N-body Schrödinger operators can be found in [5], see also references therein.

Finally, we study accumulation of eigenvalues. Here we combine our results on decay of eigenfunctions with the compactness argument due to Perry [27]. Let us remark that an attempt to describe accumulation of eigenvalues for gen-

eral elliptic selfadjoint problems in domains with cylindrical ends was made in [15]. However, as it was observed later [29], the results on accumulation of eigenvalues announced in [15] are valid only under an additional assumption on exponential decay of eigenfunctions; see [13] for the proof of other results announced in [15]. Despite that after simple modifications the approach [13,15,29] is capable to prove some results of this paper, we prefer to rely on methods similar to those we meet in the theory of Schrödinger operators [3,5,9,27,30]. First, because this demonstrates a certain similarity between the theory of Schrödinger operators and the analysis on manifolds with axial analytic asymptotically cylindrical ends. Secondly, because these methods are simpler: in contrast to [13,15,29] they do not require from the reader an extensive background in the analytic Fredholm theory [7] nor any prior knowledge of methods and results of the asymptotic theory [17,18,20]. Nonetheless, throughout the paper we make corresponding remarks every time one assertion or another can equivalently be obtained by methods of [13,15,17,18,20,29].

The structure of this paper is as follows. In Section 2 we introduce manifolds with asymptotically cylindrical and axial analytic asymptotically cylindrical ends. Section 3 presents a summary of main results of this paper and two illustrative examples. In Section 4 we give a broad overview of our approach to the complex scaling [12]. In Section 5 we study the quadratic form and localize the essential spectrum of the Laplacian deformed by means of the complex scaling and conjugated with an exponent. Then in Section 6 we prove that all non-threshold eigenfunctions of the Laplacian are of some exponential decay at infinity. Finally, in Section 7 we show that the rate of eigenfunction decay is prescribed by the distance from the corresponding eigenvalue to the next threshold, and study accumulation of eigenvalues.

2 Manifolds with axial analytic asymptotically cylindrical ends

Let Ω be a smooth compact n-dimensional manifold with smooth boundary $\partial\Omega$ or without it. Denote by Π the semi-cylinder $\mathbb{R}_+ \times \Omega$, where $\mathbb{R}_+$ is the positive semi-axis, and $\times$ stands for the Cartesian product. Consider a smooth oriented connected $n+1$-dimensional manifold $\mathcal{M}$ representable in the form $\mathcal{M} = \mathcal{M}_c \cup \Pi$, where $\mathcal{M}_c$ is a smooth compact manifold with boundary, cf. Fig. 1 and Fig. 2. We do not consider the case of a manifold $\mathcal{M}$ with compact boundary $\partial\mathcal{M}$ as it can be treated similarly to the case of a manifold without boundary. Namely, we assume that $\partial\mathcal{M} = \varnothing$ in the case $\partial\Omega = \varnothing$.

Let $\mathsf{g} \in C^\infty \mathrm{T}^*\mathcal{M}^{\otimes 2}$ be a Riemannian metric on $\mathcal{M}$. We identify the cotangent bundle $\mathrm{T}^*\Pi$ with the tensor product $\mathrm{T}^*\mathbb{R}_+ \otimes \mathrm{T}^*\Omega$ via the natural isomorphism induced by the product structure on Π. This together with the trivialization $\mathrm{T}^*\mathbb{R}_+ = \{(x, a\,dx) : x \in \mathbb{R}_+, a \in \mathbb{R}\}$ implies that any metric g can be repre-

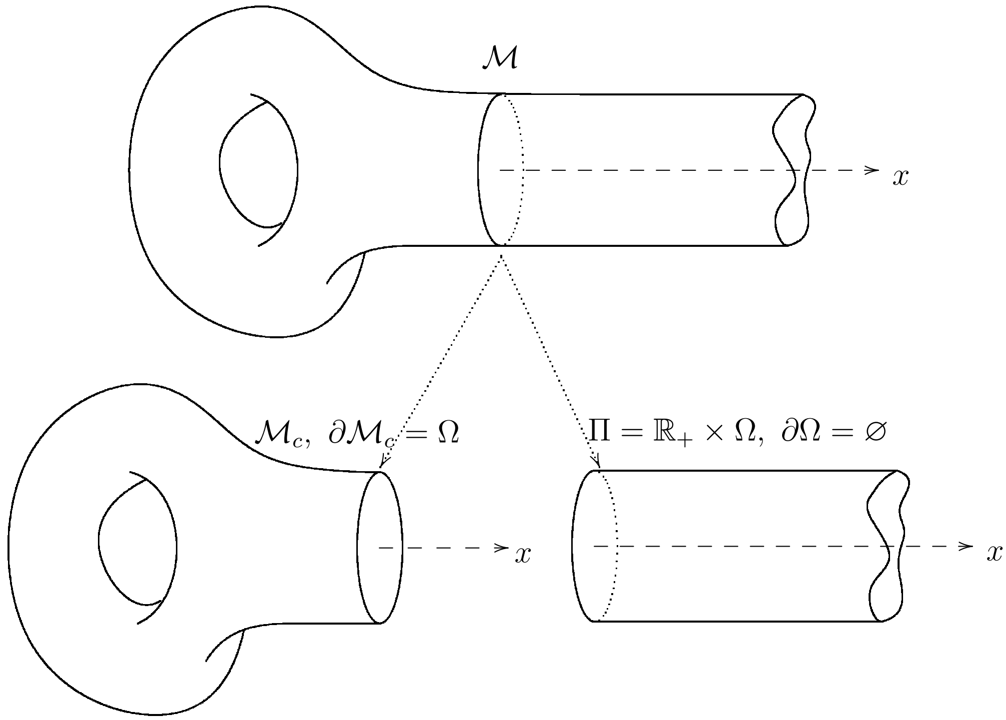

Fig. 1. Representation $\mathcal{M} = \mathcal{M}_c \cup \Pi$ of a manifold $\mathcal{M}$ without boundary.

sented on Π in the form

$$\mathfrak{g}\restriction_\Pi = \mathfrak{g}_0 dx \otimes dx + 2\mathfrak{g}_1 \otimes dx + \mathfrak{g}_2, \quad \mathfrak{g}_k(x) \in C^\infty \mathrm{T}^*\Omega^{\otimes k}. \tag{2.1}$$

Denote by $\mathbb{C}\mathrm{T}^*\Omega^{\otimes k}$ the tensor power of the complexified cotangent bundle $\mathbb{C}\mathrm{T}^*\Omega$ with the fibers $\mathbb{C}\mathrm{T}^*_{\mathrm{y}}\Omega = \mathrm{T}^*_{\mathrm{y}}\Omega \otimes \mathbb{C}$. In what follows C^m stands for sections of complexified bundles, e.g. we write $C^\infty \mathrm{T}^*\Omega^{\otimes k}$ and $C^1 \mathrm{T}^*\Omega^{\otimes k}$ instead of $C^\infty \mathbb{C}\mathrm{T}^*\Omega^{\otimes k}$ and $C^1 \mathbb{C}\mathrm{T}^*\Omega^{\otimes k}$. We equip the space $C^1 \mathrm{T}^*\Omega^{\otimes k}$ with the norm

$$\|\cdot\|_{\mathfrak{e}} = \max_{\mathrm{y}\in\Omega}\Big(|\cdot|_{\mathfrak{e}}(\mathrm{y}) + |D\cdot|_{\mathfrak{e}}(\mathrm{y})\Big), \tag{2.2}$$

where $\mathfrak{e}$ is a Riemannian metric on Ω, $|\cdot|_{\mathfrak{e}}(\mathrm{y})$ is the norm induced by $\mathfrak{e}$ in the fiber $\mathbb{C}\mathrm{T}^*_{\mathrm{y}}\Omega^{\otimes k}$, and $D : C^1\mathrm{T}^*\Omega^{\otimes k} \to C^0\mathrm{T}^*\Omega^{\otimes k+1}$ is the Levi-Civita connection on the manifold $(\Omega, \mathfrak{e})$.

Definition 2.1 *We say that* $(\mathcal{M}, \mathfrak{g})$ *is a manifold with an axial analytic asymptotically cylindrical end* $(\Pi, \mathfrak{g}\restriction_\Pi)$*, if the following conditions hold:*

i. The functions $x \mapsto \mathfrak{g}_k(x) \in C^\infty\mathrm{T}^*\Omega^{\otimes k}$ *in* (2.1) *extend by analyticity in* x *from* $\mathbb{R}_+$ *to the sector* $\mathbb{S}_\alpha = \{z \in \mathbb{C} : |\arg z| < \alpha\}$ *with some* $\alpha > 0$.
ii. As z *tends to infinity in* $\mathbb{S}_\alpha$ *the function* $\mathfrak{g}_0(z)$ *uniformly converges to* 1 *in the norm of* $C^1(\Omega)$*, the tensor field* $\mathfrak{g}_1(z)$ *uniformly converges to zero*

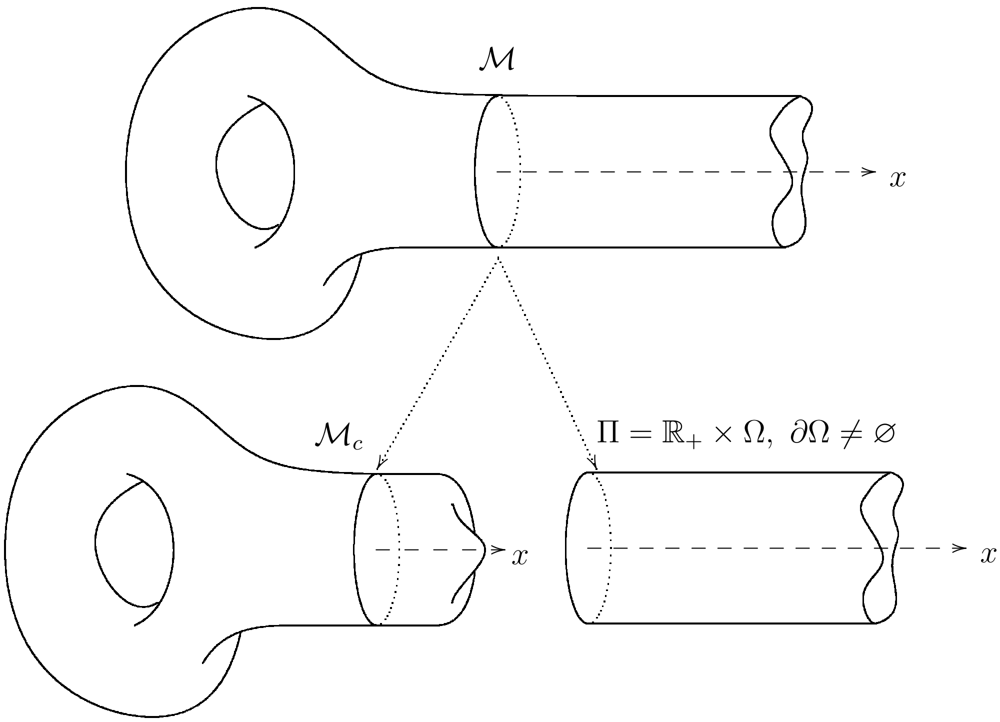

Fig. 2. Representation $\mathcal{M} = \mathcal{M}_c \cup \Pi$ of a manifold $\mathcal{M}$ with boundary.

in the norm of $C^1\mathrm{T}^*\Omega$*, and the tensor field* $\mathfrak{g}_2(z)$ *uniformly converges to a Riemannian metric* $\mathfrak{h}$ *on* Ω *in the norm of* $C^1\mathrm{T}^*\Omega^{\otimes 2}$.

In this paper we are mainly concerned in manifolds with axial analytic asymptotically cylindrical ends. However, some of our results are valid without any assumptions on the axial analytic regularity of the metric $\mathsf{g}\restriction_\Pi$. In those cases we consider general manifolds with asymptotically cylindrical ends in the sense of the following definition.

Definition 2.2 *We say that* $(\mathcal{M}, \mathsf{g})$ *is a manifold with an asymptotically cylindrical end* $(\Pi, \mathsf{g}\restriction_\Pi)$*, if for some Riemannian metric* $\mathfrak{h}$ *on* Ω *we have*

$$\|\mathfrak{g}_0(x) - 1\|_{\mathfrak{e}} + \|\mathfrak{g}_1(x)\|_{\mathfrak{e}} + \|\mathfrak{g}_2(x) - \mathfrak{h}\|_{\mathfrak{e}} \to 0 \text{ as } x \to +\infty,$$

and also

$$\max_{\mathrm{y}\in\Omega} \sum_{k=0}^{2} |\partial_x \mathfrak{g}_k(x)|_{\mathfrak{e}}(\mathrm{y}) \to 0 \text{ as } x \to +\infty,$$

where $\mathfrak{g}_k$ *are the coefficients in* (2.1)*, and* $\partial_x = d/dx$ *is the real derivative.*

Note that Definitions 2.2 and 2.1 are independent of the metric $\mathfrak{e}$ on Ω. It is a consequence of the Cauchy inequalities that any manifold with an axial analytic asymptotically cylindrical end is also a manifold with an asymptotically cylindrical end.

3 Summary of main results

Consider a manifold $(\mathcal{M}, \mathsf{g})$ with an asymptotically cylindrical end $(\Pi, \mathsf{g}\restriction_\Pi)$. We introduce the Hilbert space $L^2(\mathcal{M})$ as the completion of the set $C_c^\infty(\mathcal{M})$ with respect to the norm $\|\cdot\| = \sqrt{(\cdot,\cdot)}$, where $(\cdot,\cdot)$ is the global inner product on $(\mathcal{M}, \mathsf{g})$. Let Δ be the Laplacian on $(\mathcal{M}, \mathsf{g})$ initially defined on a core $\mathbf{C}(\Delta)$. In the case $\partial\mathcal{M} = \varnothing$ we take $\mathbf{C}(\Delta) \equiv C_c^\infty(\mathcal{M})$, while in the case $\partial\mathcal{M} \neq \varnothing$ the core $\mathbf{C}(\Delta)$ of the Neumann (resp. Dirichlet) Laplacian Δ consists of the functions $u \in C_c^\infty(\mathcal{M})$ satisfying the Neumann boundary condition $\partial_\nu u = 0$ (resp. the Dirichlet boundary condition $u\restriction_{\partial\mathcal{M}} = 0$).

Let $(\Omega, \mathfrak{h})$ be the same compact Riemannian manifold as in Definition 2.1. Recall that we exclude from consideration the case of a manifold $\mathcal{M}$ with compact boundary $\partial\mathcal{M}$, i.e. the equalities $\partial\mathcal{M} = \varnothing$ and $\partial\Omega = \varnothing$ can hold only simultaneously. If $\partial\mathcal{M} \neq \varnothing$ and Δ is the Dirichlet (resp. Neumann) Laplacian on $(\mathcal{M}, \mathsf{g})$, then by Δ_Ω we denote the Dirichlet (resp. Neumann) Laplacian on $(\Omega, \mathfrak{h})$. If $\partial\mathcal{M} = \varnothing$, then Δ_Ω is the Laplacian on the manifold $(\Omega, \mathfrak{h})$ without boundary. Let $L^2(\Omega)$ be the Hilbert space of all square summable functions on $(\Omega, \mathfrak{h})$. As is well-known, the spectrum of the operator Δ_Ω in $L^2(\Omega)$ consists of infinitely many nonnegative isolated eigenvalues. Let $\nu_1 < \nu_2 < \dots$ be the distinct eigenvalues of Δ_Ω. By definition $\{\nu_j\}_{j=1}^\infty$ is the set of thresholds of the Laplacian Δ on $(\mathcal{M}, \mathsf{g})$. Let us stress that the thresholds of the Dirichlet and the Neumann Laplacians are different.

The main results of this paper are listed in the next theorem.

Theorem 3.1 *Let Δ be the Laplacian on a manifold $(\mathcal{M}, \mathsf{g})$ with an axial analytic asymptotically cylindrical end. Then the following assertions are valid.*

1. *The eigenvalues of the selfadjoint operator Δ in $L^2(\mathcal{M})$ are of finite multiplicity and can accumulate only at the thresholds $\nu_1, \nu_2, \dots,$ and only from below.*
2. *Any eigenfunction Ψ corresponding to a non-threshold eigenvalue μ of Δ meets the estimate*

$$\|\Psi(x)\|_{L^2(\Omega)} \leqslant Ce^{\gamma x} \text{ as } x \to +\infty \tag{3.1}$$

with any negative $\gamma > -\min_{j:\nu_j>\mu}\sqrt{\nu_j - \mu}$ and an independent of x constant C. Here $x \in \mathbb{R}_+$ is the axial coordinate of the end Π, see Fig. 1 and Fig. 2, and $\min_{j:\nu_j>\mu}(\nu_j - \mu)$ is the distance from μ to the next threshold of Δ.

Similar results for N-body Schrödinger operators can be found e.g. in [5,27,30]. We complete this section with examples of manifolds with axial analytic asymptotically cylindrical ends for which the eigenvalues of the Laplacian accumulate at thresholds.

Consider a smooth compact n-dimensional Riemannian manifold $(\Omega', \mathfrak{h})$ with smooth boundary or without it. Let the infinite cylinder $\mathcal{M} = \mathbb{R} \times \Omega'$ be endowed with the metric $\mathsf{g} = dx \otimes dx + f(x)^{4/n}\mathfrak{h}$, where f is a smooth positive function on $\mathbb{R}$, such that $f(x)^{4/n} = 1+|x|^{-\delta}$ for $|x| \geqslant c > 0$ and $\delta \in (0, 2]$. Then $(\mathcal{M}, \mathsf{g})$ can be viewed as a manifold with the axial analytic asymptotically cylindrical end $(\mathbb{R}_+ \times \Omega, dx\otimes dx+f(x+c)^{4/n}\mathfrak{h})$, where Ω consists of two copies of Ω'. Let $\{\sigma_k\}_{k=1}^{\infty}$ be the eigenvalues of $\Delta_{\Omega'}$ listed with multiplicity. Separation of variables [2] shows that the Laplacian $\Delta = \frac{1}{f}(-\partial_x^2 + f''/f + f^{-4/n}\Delta_{\Omega'})f$ on $(\mathcal{M}, \mathsf{g})$ (it is the Dirichlet Laplacian in the case $\partial\mathcal{M} \neq \varnothing$) is unitary equivalent to the direct sum of the unbounded operators $-\partial_x^2 + f''/f + f^{-4/n}\sigma_k$ acting in $L^2(\mathbb{R})$. Therefore μ is an eigenvalue of Δ, if and only if $\mu - \sigma_k$ is an eigenvalue of the Schrödinger operator $-\partial_x^2 + V_k$ in $L^2(\mathbb{R})$ for some k, where $V_k = f''/f + (f^{-4/n} - 1)\sigma_k$ is the potential. The minimax principle implies that for all sufficiently large σ_k the discrete eigenvalues of $-\partial_x^2 + V_k$ accumulate at zero from below, cf. [30, Theorem XIII.6]. Thus the embedded eigenvalues of Δ accumulate at every sufficiently large threshold $\nu_j = \sigma_k$.

Let $f \in C^\infty(\mathbb{R})$, $f(s) > 0$, and $f(s) = 1 + 5|s|^{-\delta}$ for $|s| \geqslant c > 0$ and $\delta \in (0, 2]$. The domain $\mathcal{G} = \{(s,t) \in \mathbb{R}^2 : |t| \leqslant f(s)\}$ can be viewed as a manifold $(\mathcal{M}, \mathsf{g})$ with an axial analytic asymptotically cylindrical end. Indeed, we can set $\Pi = \mathbb{R}_+ \times \{[-1,1] \cup [-1,1]\}$ and define $\mathsf{g} \restriction_\Pi$ as the pullback of the Euclidean metric by the diffeomorphism $(\pm s, t) = (x + c, f(x+c)y)$ mapping Π onto the asymptotic semi-strips $\{(s,t) \in \mathcal{G} : \pm s > c\}$; here $y \in [-1,1]$ is the local coordinate on $\Omega = [-1,1] \cup [-1,1]$. Due to the axial symmetry of $\mathcal{G}$ it is possible to prove by the minimax principle that the embedded eigenvalues of the Neumann Laplacian on $(\mathcal{M}, \mathsf{g})$ accumulate at the first non-zero threshold $\nu_2 = \pi^2/4$, while the isolated and embedded eigenvalues of the Dirichlet Laplacian on $(\mathcal{M}, \mathsf{g})$ accumulate at the thresholds $\nu_1 = \pi^2/4$ and $\nu_2 = \pi^2$ correspondingly; for details we refer to [4].

Other examples of manifolds with axial analytic asymtotically cylindrical ends can be found in [12], see also [11].

4 An approach to the complex scaling

This section presents a broad overview of our approach [12] to the complex scaling on manifolds with axial analytic asymptotically cylindrical ends. The approach originates from the one developed in [9] for N-body Schrödinger operators.

We use the complex scaling $\mathbb{R}_+ \ni x \mapsto x + \lambda \mathsf{s}_R(x)$ along the axis of the semi-cylinder $\Pi = \mathbb{R}_+ \times \Omega$. Here $\mathsf{s}_R(x) = \mathsf{s}(x - R)$ is a scaling function with a sufficiently large parameter $R > 0$ and a smooth function s possessing the

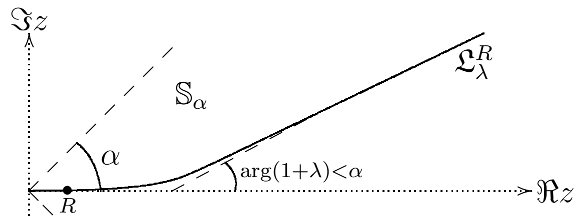

Fig. 3. The curve $\mathfrak{L}^R_\lambda = \{z \in \mathbb{C} : z = x + \lambda \mathsf{s}_R(x), x \in \mathbb{R}_+\}$ with $\lambda \in \mathcal{D}_\alpha$.

properties:

$$\begin{aligned} &\mathsf{s}(x) = 0 \text{ for all } x \leqslant 1, \\ &0 \leqslant \mathsf{s}'(x) \leqslant 1 \text{ for all } x \in \mathbb{R}, \text{ and } \mathsf{s}'(x) = 1 \text{ for large } x > 0, \end{aligned} \tag{4.1}$$

where $\mathsf{s}' = \partial \mathsf{s}/\partial x$. The scaling parameter λ takes its values in the disk

$$\mathcal{D}_\alpha = \{\lambda \in \mathbb{C} : |\lambda| < \sin\alpha < 1/\sqrt{2}\}, \tag{4.2}$$

where $\alpha < \pi/4$ is some angle for which the conditions of Definition 2.1 hold. The function $\mathbb{R}_+ \ni x \mapsto x + \lambda \mathsf{s}_R(x)$ is invertible for all real $\lambda \in (-1, 1)$, and thus defines the selfdiffeomorphism

$$\Pi \ni (x, \mathrm{y}) \mapsto \varkappa_\lambda(x, \mathrm{y}) = (x + \lambda \mathsf{s}_R(x), \mathrm{y}) \in \Pi,$$

which scales the semi-cylinder Π along its axis. We extend $\varkappa_\lambda$ to a selfdiffeomorphism of $\mathcal{M}$ by setting $\varkappa_\lambda(p) = p$ for all $p \in \mathcal{M} \setminus \Pi$. As a result we get the Riemannian manifolds $(\mathcal{M}, \mathsf{g}_\lambda)$ parametrized by $\lambda \in (-1, 1)$, where $\mathsf{g}_\lambda = \varkappa^*_\lambda \mathsf{g}$ is the pullback of the metric g by $\varkappa_\lambda$.

Let $\mathrm{T}'^*\mathbb{S}_\alpha$ be the holomorphic cotangent bundle $\{(z, c\, dz) : z \in \mathbb{S}_\alpha, c \in \mathbb{C}\}$ of the sector $\mathbb{S}_\alpha = \{z \in \mathbb{C} : |\arg z| < \alpha < \pi/4\}$, where $dz = d\Re z + i d\Im z$. Consider the tensor field

$$\mathfrak{g}_0 dz \otimes dz + 2\mathfrak{g}_1 \otimes dz + \mathfrak{g}_2 \in C^\infty(\mathrm{T}'^*\mathbb{S}_\alpha \otimes \mathrm{T}^*\Omega)^{\otimes 2} \tag{4.3}$$

with the analytic coefficients $\mathbb{S}_\alpha \ni z \mapsto \mathfrak{g}_k(z) \in C^\infty \mathrm{T}^*\Omega^{\otimes k}$, cf. Definition 2.1. For all $\lambda \in \mathcal{D}_\alpha$ the complex scaling defines the embedding

$$\mathrm{T}^*\mathbb{R}_+ \ni \{x, a\, dx\} \mapsto \{x + \lambda \mathsf{s}_R(x), a(1 + \lambda \mathsf{s}'_R(x))^{-1} dz\} \in \mathrm{T}'^*\mathbb{S}_\alpha, \tag{4.4}$$

where $|1 + \lambda \mathsf{s}'_R(x)| > 1 - 1/\sqrt{2}$, see Fig. 3. We identify the bundle $\mathrm{T}^*\Pi$ with the product $\mathrm{T}^*\mathbb{R}_+ \otimes \mathrm{T}^*\Omega$ via the product structure on $\Pi = \mathbb{R}_+ \times \Omega$. The embedding (4.4) together with (4.3) induces the tensor field

$$\begin{aligned} \mathsf{g}_\lambda \restriction_\Pi = \mathfrak{g}^R_{0,\lambda} dx \otimes dx + 2\mathfrak{g}^R_{1,\lambda} \otimes dx + \mathfrak{g}^R_{2,\lambda} \in C^\infty \mathrm{T}^*\Pi^{\otimes 2}, \\ \mathfrak{g}^R_{k,\lambda}(x) = (1 + \lambda \mathsf{s}'_R(x))^{2-k} \mathfrak{g}_k(x + \lambda \mathsf{s}_R(x)), \end{aligned} \tag{4.5}$$

where $\mathfrak{g}^R_{k,\lambda}(x) \in C^\infty \mathrm{T}^*\Omega^{\otimes k}$ are smooth in $x \in \mathbb{R}_+$ and analytic in $\lambda \in \mathcal{D}_\alpha$ coefficients. Since $\operatorname{supp} \mathsf{s}_R \cap (0, R) = \varnothing$, the equality $\mathsf{g}_\lambda \restriction_{(0,R)\times\Omega} = \mathsf{g} \restriction_{(0,R)\times\Omega}$

holds for all $\lambda \in \mathcal{D}_\alpha$, cf. (2.1) and (4.5). Thanks to this we can smoothly extend $\mathsf{g}_\lambda \upharpoonright_\Pi$ to $\mathcal{M}$ by setting $\mathsf{g}_\lambda \upharpoonright_{\mathcal{M}\setminus\Pi} = \mathsf{g} \upharpoonright_{\mathcal{M}\setminus\Pi}$. As a result we obtain an analytic function

$$\mathcal{D}_\alpha \ni \lambda \mapsto \mathsf{g}_\lambda \in C^\infty \mathrm{T}^*\mathcal{M}^{\otimes 2}.$$

We consider the tensor field g_λ with $\lambda \in \mathcal{D}_\alpha$ as a deformation of the metric g on $\mathcal{M}$ by means of the complex scaling. Clearly, for $\lambda \in \mathcal{D}_\alpha \cap \mathbb{R}$ we have $\mathsf{g}_\lambda = \varkappa_\lambda^* \mathsf{g}$, and $\mathsf{g}_0 \equiv \mathsf{g}$. By analyticity in λ we conclude that g_λ is a symmetric tensor field. The Schwarz reflection principle gives $\overline{\mathsf{g}_\lambda} = \mathsf{g}_{\overline{\lambda}}$, where the bar stands for the complex conjugation. It must be stressed that the tensor field g_λ with $\lambda \neq 0$ depends on R, however we do not indicate this for brevity of notations. As shown in [12], the condition ii in Definition 2.1 implies that the tensor fields g_λ with $\lambda \in \mathcal{D}_\alpha$ are non-degenerate as $R > 0$ is sufficiently large.

It is well known that Riemannian metrics induce musical isomorphisms between the tangent and cotangent bundles. Similarly, the non-degenerate tensor field g_λ induces a musical fiber isomorphism ${}^\lambda\sharp : \mathbb{C}\mathrm{T}^*\mathcal{M} \to \mathbb{C}\mathrm{T}\mathcal{M}$ between the complexified bundles. For every $p \in \mathcal{M}$ the tensor field $\mathsf{g}_\lambda \in C^\infty \mathrm{T}^*\mathcal{M}^{\otimes 2}$ naturally defines a non-degenerate sesquilinear form $\mathsf{g}_\lambda^p[\cdot,\cdot]$ on $\mathbb{C}\mathrm{T}_p\mathcal{M}$. The isomorphism ${}^\lambda\sharp$ acts by the rule

$$\mathbb{C}\mathrm{T}_p^*\mathcal{M} \ni \xi \mapsto {}^\lambda\sharp\xi \in \mathbb{C}\mathrm{T}_p\mathcal{M},$$

where ${}^\lambda\sharp\xi$ is a unique vector satisfying the equality $\xi\overline{\eta} = \mathsf{g}_\lambda^p[{}^\lambda\sharp\xi, \eta]$ for all $\eta \in \mathbb{C}\mathrm{T}_p\mathcal{M}$. We extend the form $\mathsf{g}_\lambda^p[\cdot,\cdot]$ to the pairs $(\xi,\omega) \in \mathbb{C}\mathrm{T}_p^*\mathcal{M} \times \mathbb{C}\mathrm{T}_p^*\mathcal{M}$ by setting

$$\mathsf{g}_\lambda^p[\xi,\omega] = \mathsf{g}_\lambda^p[{}^\lambda\sharp\,\xi, {}^{\overline{\lambda}}\sharp\,\omega], \quad \lambda \in \mathcal{D}_\alpha.$$

The function $\mathcal{D}_\alpha \ni \lambda \mapsto \mathsf{g}_\lambda^p[\xi,\omega]$ is analytic. If $\lambda \in \mathcal{D}_\alpha$ is real, then $\mathsf{g}_\lambda^p[\cdot,\cdot]$ is the positive Hermitian form corresponding to the Riemannian metric g_λ on $\mathcal{M}$. In particular, $\mathsf{g}_0^p[\cdot,\cdot] \equiv \mathsf{g}^p[\cdot,\cdot]$. However, for a non-real $\lambda \in \mathcal{D}_\alpha$ we have $\overline{\mathsf{g}_\lambda^p[\xi,\omega]} = \mathsf{g}_{\overline{\lambda}}^p[\omega,\xi]$, and the form $\mathsf{g}_\lambda^p[\cdot,\cdot]$ is not Hermitian. Nonetheless, on the differential one-forms the sesquilinear form $\mathsf{g}_\lambda^p[\cdot,\cdot]$ is sectorial and relatively bounded [12, Lemma 4.2]. More precisely, there exist some independent of $p \in \mathcal{M}$ and $\lambda \in \mathcal{D}_\alpha$ angle $\vartheta < \pi/2$ and constant $\delta > 0$, such that

$$|\arg \mathsf{g}_\lambda^p[\xi,\xi]| \leqslant \vartheta, \quad \delta\mathsf{g}^p[\xi,\xi] \leqslant \Re\mathsf{g}_\lambda^p[\xi,\xi] \leqslant \delta^{-1}\mathsf{g}^p[\xi,\xi] \quad \forall \xi \in \mathbb{C}\mathrm{T}_p^*\mathcal{M}. \tag{4.6}$$

By dvol_λ with real $\lambda \in \mathcal{D}_\alpha$ we denote the volume form on the Riemannian manifold $(\mathcal{M}, \mathsf{g}_\lambda)$. The equality $\varrho_\lambda\,\mathrm{dvol}_\lambda = \mathrm{dvol}_0$ defines a function $\varrho_\lambda \in C^\infty(\mathcal{M})$. Due to the properties of g_λ it turns out that $\varrho_\lambda \in C^\infty(\mathcal{M})$ is an analytic function of $\lambda \in \mathcal{D}_\alpha$ obeying the estimates

$$0 < c \leqslant |\varrho_\lambda(p)| \leqslant 1/c, \quad \mathsf{g}^p[d\varrho_\lambda, d\varrho_\lambda] < 1/c, \quad p \in \mathcal{M}, \tag{4.7}$$

where c is independent of p and λ. We introduce the deformed volume form

$$\mathrm{dvol}_\lambda := \frac{1}{\varrho_\lambda}\,\mathrm{dvol}_0, \quad \lambda \in \mathcal{D}_\alpha,$$

and the deformed global inner product

$$(\xi,\omega)_\lambda = \int_{\mathcal{M}} \mathsf{g}_\lambda[\xi,\omega]\;\mathrm{dvol}_\lambda, \quad \xi,\omega \in C_c^\infty \mathrm{T}^*\mathcal{M}^{\otimes k}.$$

Let us stress that for non-real $\lambda \in \mathcal{D}_\alpha$ the deformed volume form is complex-valued, and the deformed inner product $(\xi,\omega)_\lambda = \overline{(\omega,\xi)_{\overline{\lambda}}}$ is not Hermitian.

Let $L^2\mathrm{T}^*\mathcal{M}^{\otimes k}$ be the completion of the set $C_c^\infty\mathrm{T}^*\mathcal{M}^{\otimes k}$ with respect to the global inner product $(\cdot,\cdot) \equiv (\cdot,\cdot)_0$. The estimates (4.6) together with the bounds on ϱ_λ imply that the deformed global inner product $(\cdot,\cdot)_\lambda$ extends to a uniformly bounded non-degenerate form in $L^2\mathrm{T}^*\mathcal{M}^{\otimes k}$ with $k=0,1$; i.e. for some independent of $\lambda \in \mathcal{D}_\alpha$ constant $c>0$ we have

$$|(\xi,\omega)_\lambda|^2 \leqslant c(\xi,\xi)(\omega,\omega), \quad \forall \xi,\omega \in L^2\mathrm{T}^*\mathcal{M}^{\otimes k}, k=0,1, \tag{4.8}$$

and for any nonzero $\xi \in L^2\mathrm{T}^*\mathcal{M}^{\otimes k}$ there exists $\omega \in L^2\mathrm{T}^*\mathcal{M}^{\otimes k}$, such that $(\xi,\omega)_\lambda \neq 0$. If $\lambda \in \mathcal{D}_\alpha$ is real, then $(\cdot,\cdot)_\lambda$ coincides with the global inner product on the Riemannian manifold $(\mathcal{M},\mathsf{g}_\lambda)$, and $\sqrt{(\cdot,\cdot)_\lambda}$ is an equivalent norm in $L^2\mathrm{T}^*\mathcal{M}^{\otimes k}$, $k=0,1$.

Let ${}^\lambda\Delta : C_c^\infty(\mathcal{M}) \to C_c^\infty(\mathcal{M})$ with $\lambda \in \mathbb{R}\cap\mathcal{D}_\alpha$ be the Laplacian on the Riemannian manifold $(\mathcal{M},\mathsf{g}_\lambda)$. In the case $\partial\mathcal{M} \neq \varnothing$ we also consider the operator ${}^\lambda\partial_\nu : C_c^\infty(\mathcal{M}) \to C_c^\infty(\partial\mathcal{M})$ of the Neumann boundary condition on $(\mathcal{M},\mathsf{g}_\lambda)$. On a manifold with an axial analytic asymptotically cylindrical end the operators ${}^\lambda\Delta$ and ${}^\lambda\partial_\nu$ extend by analyticity from $\mathbb{R}\cap\mathcal{D}_\alpha$ to all $\lambda \in \mathcal{D}_\alpha$, see [12]. Moreover, by taking a sufficiently large parameter $R>0$ we arrange the complex scaling so that the differential operator ${}^\lambda\Delta$ on $\mathcal{M}$ is strongly elliptic, and in the case $\partial\mathcal{M} \neq \varnothing$ the pair $\{{}^\lambda\Delta, {}^\lambda\partial_\nu\}$ obeys the Shapiro-Lopatinskiĭ condition on $\partial\mathcal{M}$; see [12, Lemmas 6.2 and 6.3]. Recall that for a strongly elliptic operator with the Dirichlet boundary condition the Shapiro-Lopatinskiĭ condition is always fulfilled, e.g. [18,19].

Let $\{\mathscr{U}_j,\kappa_j\}$ be a finite atlas on $(\Omega,\mathfrak{h})$, and let $y \in \mathbb{R}^n$ be a system of local coordinates in a neighborhood $\mathscr{U}_j$. If $\partial\Omega\cap\mathscr{U}_j \neq \varnothing$, then we assume in addition that all y in the image of the set $\partial\Omega\cap\mathscr{U}_j$ under the diffeomorphism κ_j are of the form $y=(y',y_n)$ with $y' \in \mathbb{R}^{n-1}$ and $y_n \geqslant 0$, and the set $\partial\Omega\cap\mathscr{U}_j$ is defined by the equality $y_n=0$. In the coordinates (x,y) on Π, where $x \in \mathbb{R}_+$ is the axial coordinate, the operators ${}^\lambda\Delta$ and ${}^\lambda\partial_\nu$ with $\lambda \in \mathcal{D}_\alpha$ admit the local

representations

$$
\begin{aligned}
{}^{\lambda}\Delta &= -\frac{1}{\sqrt{\det \mathbf{g}_\lambda}} \nabla_{xy} \cdot \sqrt{\det \mathbf{g}_\lambda}\; \mathbf{g}_\lambda^{-1} \nabla_{xy}, \\
{}^{\lambda}\partial_\nu &= \Big(0, \dots, 0, 1/\sqrt{\mathbf{g}^{-1}_{\lambda, nn}}\Big) \mathbf{g}_\lambda^{-1} \restriction_{y_n = 0} \nabla_{xy} \quad \text{ if } \quad \mathscr{U}_j \cap \partial\Omega \neq \varnothing.
\end{aligned}
\tag{4.9}
$$

Here the complex symmetric matrix $\mathbf{g}_\lambda$ corresponds to the representation of the tensor field g_λ in the coordinates (x, y), and $\nabla_{xy} \equiv (\partial_x, \partial_{y_1}, \dots, \partial_{y_n})^\intercal$. As shown in [12, Lemma 4.1], the inverse matrix $\mathbf{g}_\lambda^{-1}$ possesses the property

$$
\sum_{|r|+q \leqslant 1} \|\partial_x^q \partial_y^r (\mathbf{g}_\lambda^{-1}(x, y) - \operatorname{diag}\{(1+\lambda)^{-2}, \mathbf{h}^{-1}(y)\})\|_2 \to 0 \text{ as } x \to +\infty \tag{4.10}
$$

uniformly in λ and y, where the matrix $\mathbf{h}(y)$ corresponds to the representation of the metric $\mathfrak{h}$ in the local coordinates, and $\|\mathbf{g}\|_2 = \sqrt{\sum_{\ell,m=0}^n |\mathbf{g}_{\ell m}|^2}$ is the matrix norm. Let us remark that ${}^{\lambda}\Delta = {}^{0}\Delta$ and ${}^{\lambda}\partial_\nu = {}^{0}\partial_\nu$ on $\mathcal{M} \setminus \Pi$. We also note that in the case of a general manifold $(\mathcal{M}, \mathsf{g})$ with an asymptotically cylindrical end the representations (4.9) and the property (4.10) remain valid for $\lambda = 0$ (and even for all real $\lambda \in (-1, 1)$) as it follows from Definition 2.2.

Consider ${}^{\lambda}\Delta$ as an unbounded operator in the Hilbert space $L^2(\mathcal{M})$, initially defined on a dense in $L^2(\mathcal{M})$ core $\mathbf{C}({}^{\lambda}\Delta)$.

Definition 4.1 *In the case $\partial\mathcal{M} = \varnothing$ we take $\mathbf{C}({}^{\lambda}\Delta) \equiv C_c^\infty(\mathcal{M})$. In the case of the Neumann (resp. Dirichlet) Laplacian Δ the core $\mathbf{C}({}^{\lambda}\Delta)$ consists of the functions $u \in C_c^\infty(\mathcal{M})$ satisfying the deformed Neumann boundary condition ${}^{\lambda}\partial_\nu u = 0$ (resp. the Dirichlet boundary condition $u \restriction_{\partial\mathcal{M}} = 0$).*

The operator ${}^{\lambda}\Delta$ with $\lambda \in \mathcal{D}_\alpha$ is a deformation of the Laplacian $\Delta \equiv {}^{0}\Delta$ by means of the complex scaling. In general, the operator ${}^{\lambda}\partial_\nu$ and the core $\mathbf{C}({}^{\lambda}\Delta)$ depend on the scaling parameter $\lambda \in \mathcal{D}_\alpha$.

Let $d : C_c^\infty(\mathcal{M}) \to C_c^\infty \mathrm{T}^*\mathcal{M}$ be the exterior derivative. We introduce the sesquilinear quadratic form

$$
\mathsf{q}_\lambda[u, v] = \Big(du, d(\overline{\varrho_\lambda} v)\Big)_\lambda, \quad u, v \in \mathbf{C}(\mathsf{q}), \quad \lambda \in \mathcal{D}_\alpha,
$$

on a core $\mathbf{C}(\mathsf{q})$.

Definition 4.2 *In the case $\partial\mathcal{M} = \varnothing$, and also in the case of the Neumann Laplacian, we take $\mathbf{C}(\mathsf{q}) \equiv C_c^\infty(\mathcal{M})$. In the case of the Dirichlet Laplacian the core $\mathbf{C}(\mathsf{q})$ consists of the functions $u \in C_c^\infty(\mathcal{M})$ with $u \restriction_{\partial\mathcal{M}} = 0$.*

As it follows from the Green identity

$$
(du, dv)_\lambda = ({}^{\lambda}\Delta u, v)_\lambda, \quad u \in \mathbf{C}({}^{\lambda}\Delta), v \in \mathbf{C}(\mathsf{q}), \tag{4.11}
$$

to the unbounded operator ${}^\lambda\Delta$ in the Hilbert space $L^2(\mathcal{M})$ there corresponds the quadratic form $\mathsf{q}_\lambda[\cdot,\cdot]$. Clearly, $\mathsf{q}_0[\cdot,\cdot]=(d\cdot,d\cdot)$ is the nonnegative quadratic form of the Laplacian Δ. Below we formulate a result from [12], for the proof we refer to [12, Proposition 5.3].

Proposition 4.3 *Assume that $(\mathcal{M},\mathsf{g})$ is a manifold with an axial analytic asymptotically cylindrical end. Let $\mathbf{D}(\mathsf{q})$ be the Hilbert space, introduced as the completion of the core $\mathbf{C}(\mathsf{q})$ with respect to the norm $\sqrt{(du,du)+\|u\|^2}$. Then the following assertions hold.*

i. The unbounded quadratic form $\mathsf{q}_\lambda[\cdot,\cdot]$ with the domain $\mathbf{D}(\mathsf{q})$ is densely defined and closed for all $\lambda\in\mathcal{D}_\alpha$.

ii. The form $\mathsf{q}_\lambda[\cdot,\cdot]$ is sectorial and relatively bounded. More precisely, for all $\lambda\in\mathcal{D}_\alpha$ and $u\in\mathbf{D}(\mathsf{q})$ the estimates

$$|\arg(\mathsf{q}_\lambda[u,u]+a\|u\|^2)|\leqslant\vartheta<\pi/2,$$

$$b(du,du)\leqslant\Re\mathsf{q}_\lambda[u,u]+a\|u\|^2,\quad \Re\mathsf{q}_\lambda[u,u]\leqslant b^{-1}\Big((du,du)+\|u\|^2\Big)$$

hold with some constants ϑ and $a,b>0$, which are independent of λ and u.

iii. For any $u\in\mathbf{D}(\mathsf{q})$ the function $\mathcal{D}_\alpha\ni\lambda\mapsto\mathsf{q}_\lambda[u,u]$ is analytic.

5 Conjugated operator and its essential spectrum

In order to study exponential decay of eigenfunctions we consider the operator ${}^\lambda\Delta$ conjugated with an exponent and use the dilation analytic techniques, similar to those we meet in the theory of N-body Schrödinger operators, e.g. [30, Chapter XIII.11].

Let s be a smooth function on the semi-cylinder Π, which depends only on the axial variable $x\in\mathbb{R}_+$ and possesses the properties (4.1). We extend s to a smooth function on $\mathcal{M}$ by setting $\mathsf{s}\restriction_{\mathcal{M}\setminus\Pi}\equiv 0$. Consider the conjugated operator ${}^\lambda\Delta_\beta=e^{-\beta\mathsf{s}}\,{}^\lambda\Delta\,e^{\beta\mathsf{s}}$ with the parameter $\beta\in\mathbb{C}$, where $e^{\beta\mathsf{s}}$ is the operator of multiplication by the exponent. The unbounded operator ${}^\lambda\Delta_\beta$ in $L^2(\mathcal{M})$ is initially defined on the dense in $L^2(\mathcal{M})$ core

$$\mathbf{C}({}^\lambda\Delta_\beta)=\{u: e^{\beta\mathsf{s}}u\in\mathbf{C}({}^\lambda\Delta)\};\tag{5.1}$$

here $\mathbf{C}({}^\lambda\Delta)$ is the same as in Definition 4.1. The core $\mathbf{C}({}^\lambda\Delta_\beta)$ depends on the parameters $\beta\in\mathbb{C}$ and $\lambda\in\mathcal{D}_\alpha$. From (4.11) and (5.1) we get

$$(e^{-\beta\mathsf{s}}\,{}^\lambda\Delta\,e^{\beta\mathsf{s}}u,v)=\Big(d(e^{\beta\mathsf{s}}u),d(\overline{e^{-\beta\mathsf{s}}\varrho_\lambda}v)\Big)_\lambda,\quad u\in\mathbf{C}({}^\lambda\Delta_\beta),v\in\mathbf{C}(\mathsf{q}).$$

Thus to the operator ${}^{\lambda}\Delta_\beta$ there corresponds the quadratic form

$$\mathsf{q}_\lambda^\beta[u,v] = \Big(d(e^{\beta\mathsf{s}}u), d(\overline{e^{-\beta\mathsf{s}}}\varrho_\lambda v)\Big)_\lambda, \quad u, v \in \mathbf{C}(\mathsf{q}).$$

Lemma 5.1 *Assume that $(\mathcal{M},\mathsf{g})$ is a manifold with an axial analytic asymptotically cylindrical end. Then the difference $\mathsf{q}_\lambda^\beta[\cdot,\cdot] - \mathsf{q}_\lambda[\cdot,\cdot]$ has an arbitrarily small uniform in $\lambda \in \mathcal{D}_\alpha$ relative bound with respect to the form $\mathsf{q}_\lambda[\cdot,\cdot]$. More precisely, for all $\lambda \in \mathcal{D}_\alpha$ and $u \in \mathbf{C}(\mathsf{q})$ the estimate*

$$|\mathsf{q}_\lambda^\beta[u,u] - \mathsf{q}_\lambda[u,u]| \leqslant \varepsilon|\mathsf{q}_\lambda[u,u]| + C(|\beta|,\varepsilon)\|u\|^2$$

is valid, where $\varepsilon > 0$ is arbitrarily small, and the constant $C(|\beta|,\varepsilon)$ depends on $|\beta|$ and ε, but not on u or λ.

If $(\mathcal{M},\mathsf{g})$ is a general manifold with an asymptotically cylindrical end, then the assertion remains valid for $\lambda = 0$.

PROOF. Let $u \in \mathbf{C}(\mathsf{q}_0)$. After simple calculations we obtain the equality

$$\begin{aligned}\mathsf{q}_\lambda^\beta[u,u] - \mathsf{q}_\lambda[u,u] &= \beta\Big(u\, d\mathsf{s}, d(\overline{\varrho_\lambda}u)\Big)_\lambda \\ &\quad - \beta\Big(\varrho_\lambda\, du, u\, d\mathsf{s}\Big)_\lambda - \beta^2\Big(\varrho_\lambda u\, d\mathsf{s}, u\, d\mathsf{s}\Big)_\lambda.\end{aligned} \tag{5.2}$$

We will rely on (4.8). Then for an arbitrarily small $\epsilon > 0$ the first term in the right hand side of (5.2) meets the estimates

$$\begin{aligned}\Big|\Big(u\, d\mathsf{s}, d(\overline{\varrho_\lambda}u)\Big)_\lambda\Big| &\leqslant c\Big(u\, d\mathsf{s}, u\, d\mathsf{s}\Big)^{1/2}\Big(d(\overline{\varrho_\lambda}u), d(\overline{\varrho_\lambda}u)\Big)^{1/2} \\ &\leqslant c\epsilon\Big(d(\overline{\varrho_\lambda}u), d(\overline{\varrho_\lambda}u)\Big) + c\epsilon^{-1}\Big(u\, d\mathsf{s}, u\, d\mathsf{s}\Big) \\ &\leqslant 2c\epsilon(u\, d\overline{\varrho_\lambda}, u\, d\overline{\varrho_\lambda}) + 2c\epsilon(|\varrho_\lambda|^2 du, du) + c\epsilon^{-1}\Big(u\, d\mathsf{s}, u\, d\mathsf{s}\Big).\end{aligned}$$

For the summands in the last line we have

$$(u\, d\overline{\varrho_\lambda}, u\, d\overline{\varrho_\lambda}) \leqslant \sup_{p\in\mathcal{M}} \mathsf{g}^p[d\varrho_\lambda, d\varrho_\lambda]\|u\|^2, \quad (|\varrho_\lambda|^2 du, du) \leqslant \sup_{p\in\mathcal{M}} |\varrho_\lambda(p)|^2(du, du),$$

$$\Big(u\, d\mathsf{s}, u\, d\mathsf{s}\Big) \leqslant \sup_{p\in\mathcal{M}} \mathsf{g}^p[d\mathsf{s}, d\mathsf{s}]\|u\|^2.$$

The second term in the right hand side of (5.2) obeys the estimate

$$\Big|\Big(\varrho_\lambda\, du, u\, d\mathsf{s}\Big)_\lambda\Big| \leqslant c\epsilon\Big(|\varrho_\lambda|^2\, du, du\Big) + c\epsilon^{-1}\Big(u\, d\mathsf{s}, u\, d\mathsf{s}\Big).$$

Finally, for the last term in (5.2) we get

$$\Big|\Big(\varrho_\lambda u\, d\mathsf{s}, u\, d\mathsf{s}\Big)_\lambda\Big| \leqslant c\Big(\varrho_\lambda u\, d\mathsf{s}, \varrho_\lambda u\, d\mathsf{s}\Big)^{1/2}\Big(u\, d\mathsf{s}, u\, d\mathsf{s}\Big)^{1/2}$$

$$\leqslant c\Big(\sup_{p\in\mathcal{M}}|\varrho_\lambda(p)|^2\Big)^{1/2}\sup_{p\in\mathcal{M}}\mathsf{g}^p[d\mathsf{s},d\mathsf{s}]\|u\|^2.$$

Observe that $d\mathsf{s}\restriction_{\mathcal{M}\setminus\Pi}=0$ and $d\mathsf{s}\restriction_{\Pi}=\mathsf{s}'\,dx$, where $\mathsf{s}'(p)=\mathsf{s}'(x)\leqslant 1$. Due to stabilization of the metric g to the product metric $dx\otimes dx+\mathfrak{h}$ at infinity, we have $\sup_{p\in\mathcal{M}}\mathsf{g}^p[d\mathsf{s},d\mathsf{s}]\leqslant C$. Now the assertion follows from the obtained estimates combined with (4.7), (5.2), and Proposition 4.3.ii. □

Proposition 5.2 *Assume that* $(\mathcal{M},\mathsf{g})$ *is a manifold with an axial analytic asymptotically cylindrical end. Let* $\lambda\in\mathcal{D}_\alpha$ *and* $\beta\in\mathbb{C}$. *Then the following assertions hold.*

i. The unbounded sesquilinear form $\mathsf{q}^\beta_\lambda[\cdot,\cdot]$ *with the domain* $\mathbf{D}(\mathsf{q})$ *is densely defined and closed.*
ii. The form $\mathsf{q}^\beta_\lambda[\cdot,\cdot]$ *is sectorial and relatively bounded. More precisely, for all* $u\in\mathbf{D}(\mathsf{q})$ *the estimates*

$$|\arg(\mathsf{q}^\beta_\lambda[u,u]+a\|u\|^2)|\leqslant\vartheta,$$

$$b(du,du)\leqslant\Re\mathsf{q}^\beta_\lambda[u,u]+a\|u\|^2,\quad \Re\mathsf{q}^\beta_\lambda[u,u]\leqslant b^{-1}\Big((du,du)+\|u\|^2\Big)$$

hold with some angle $\vartheta<\pi/2$ *and some positive constants* a *and* b, *which may depend on* β *and* ϑ, *but not on* λ *or* u.
iii. For any $u\in\mathbf{D}(\mathsf{q})$ *and* $\lambda\in\mathcal{D}_\alpha$ *the function* $\mathbb{C}\ni\beta\mapsto\mathsf{q}^\beta_\lambda[u,u]$ *is analytic.*

In the case of a general manifold with an asymptotically cylindrical end the assertions remain valid for $\lambda=0$.

PROOF. The first two assertions are direct consequences of Lemma 5.1 and Proposition 4.3; all necessary basic facts from the theory of sesquilinear forms can be found e.g. in [16, Chapter VI]. The last assertion is an immediate consequence of the equality (5.2). □

As is known [16, Chapter VI.2.1], there is a one-to-one correspondence between the set of all densely defined closed sectorial sesquilinear forms and the set of all m-sectorial operators. (Here and elsewhere m-sectorial means that the numerical range $\{(\mathsf{A}u,u):u\in\mathbf{D}(\mathsf{A})\}$ and the spectrum of a closed operator A with domain $\mathbf{D}(\mathsf{A})$ lie in the sector $\{z\in\mathbb{C}:|\arg(z+a)|\leqslant\vartheta\}$ with some $\vartheta<\pi/2$ and $a>0$.) Thus Proposition 5.2 implies that the Friedrichs extension of the operator ${}^\lambda\Delta_\beta$, initially defined on the dense in $L^2(\mathcal{M})$ core $\mathbf{C}({}^\lambda\Delta_\beta)$, is m-sectorial. In particular, the Friedrichs extension of the Laplacian $\Delta\equiv{}^0\Delta_0$ is a nonnegative selfadjoint operator.

Consider the domain $\mathbf{D}({}^\lambda\Delta_\beta)$ of the m-sectorial operator ${}^\lambda\Delta_\beta$ as a Hilbert space introduced as the completion of the core (5.1) with respect to the graph

norm $\sqrt{\|\cdot\|^2+\|{}^\lambda\Delta_\beta\cdot\|^2}$. We say that μ is a point of the essential spectrum $\sigma_{ess}({}^\lambda\Delta_\beta)$, if the bounded operator

$$ {}^\lambda\Delta_\beta-\mu:\mathbf{D}({}^\lambda\Delta_\beta)\to L^2(\mathcal{M}) \tag{5.3}$$

is not Fredholm. (Recall that a bounded linear operator is said to be Fredholm, if its kernel and cokernel are finite-dimensional, and the range is closed.) In the next proposition we localize the essential spectrum of the operator ${}^\lambda\Delta_\beta$. For $\beta=0$ the proposition was already proven in [12, Theorem 6.1], the general case $\beta\in\mathbb{C}$ is very similar. In the proof we use methods of the theory of non-homogeneous elliptic boundary value problems [19,20], see also [17,18].

Proposition 5.3 *Assume that* $(\mathcal{M},\mathsf{g})$ *is a manifold with an axial analytic asymptotically cylindrical end. Let* $\lambda\in\mathcal{D}_\alpha$ *and* $\beta\in\mathbb{C}$. *Then* $\mu\in\sigma_{ess}({}^\lambda\Delta_\beta)$, *if and only if the equality*

$$\nu_j-\mu=(1+\lambda)^{-2}(\beta+i\xi)^2 \tag{5.4}$$

holds for some $j\in\mathbb{N}$ *and some* $\xi\in\mathbb{R}$, *where* $\{\nu_j\}_{j=1}^\infty$ *is the set of thresholds of the Laplacian* Δ *on* $(\mathcal{M},\mathsf{g})$.

The spectrum $\sigma_{ess}({}^\lambda\Delta_\beta)$ *is depicted on Fig. 4. In the case* $\beta=0$ *the parabolas of the essential spectrum collapse to the dashed rays originating from every threshold* ν_j, *and we obtain the essential spectrum* $\sigma_{ess}({}^\lambda\Delta)$ *of* ${}^\lambda\Delta\equiv{}^\lambda\Delta_0$.

In the case of a general manifold with an asymptotically cylindrical end the assertion remains valid for $\lambda=0$.

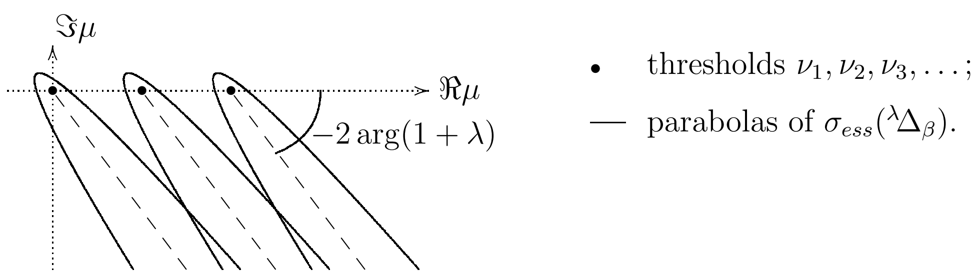

Fig. 4. Essential spectrum of the operator ${}^\lambda\Delta_\beta$ for $\Im\lambda>0$ and $\beta\gtrless 0$.

PROOF. As it was already mentioned, by taking a sufficiently large R in the function $\mathsf{s}_R(x)=\mathsf{s}(x-R)$, we arrange the complex scaling so that the deformation ${}^\lambda\Delta$ is a strongly elliptic operator on $\mathcal{M}$, and the pair $\{{}^\lambda\Delta,{}^\lambda\partial_\nu\}$ meets the Shapiro-Lopatinskiĭ condition on $\partial\mathcal{M}$, if $\partial\mathcal{M}\neq\varnothing$. The principal symbols of the operators ${}^\lambda\Delta_\beta$ and ${}^\lambda\Delta$ are coincident, as well as the principal symbols of $e^{-\beta\mathsf{s}}\,{}^\lambda\partial_\nu\,e^{\beta\mathsf{s}}$ and ${}^\lambda\partial_\nu$. Hence the differential operator ${}^\lambda\Delta_\beta$ is strongly elliptic on $\mathcal{M}$, and in the case $\partial\mathcal{M}\neq\varnothing$ the pair $\{{}^\lambda\Delta_\beta,e^{-\beta\mathsf{s}}\,{}^\lambda\partial_\nu\,e^{\beta\mathsf{s}}\}$ meets

the Shapiro-Lopatinskiĭ condition on $\partial\mathcal{M}$. All other difficulties related to the appearance of the deformed operator of the Neumann boundary conditions $e^{-\beta \mathsf{s}}\,{}^\lambda\partial_\nu\, e^{\beta\mathsf{s}}$ on $\partial\mathcal{M}$ can be handled exactly in the same way as in [12, Theorem 6.1]. For this reason here we consider only the case of the Dirichlet Laplacian on a manifold $(\mathcal{M}, \mathsf{g})$ with non-compact boundary $\partial\mathcal{M}$. The case $\partial\mathcal{M} = \varnothing$ is similar. As in [11,12] we will rely on the Peetre's lemma:

Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ be Banach spaces, where $\mathcal{X}$ is compactly embedded into $\mathcal{Z}$. Furthermore, let $\mathcal{L}$ be a linear continuous operator from $\mathcal{X}$ to $\mathcal{Y}$. Then the next two assertions are equivalent: (i) the range of $\mathcal{L}$ is closed in $\mathcal{Y}$ and $\dim\ker\mathcal{L} < \infty$, *(ii) there exists a constant C, such that*

$$\|u\|_{\mathcal{X}} \leqslant C(\|\mathcal{L}u\|_{\mathcal{Y}} + \|u\|_{\mathcal{Z}}) \quad \forall u \in \mathcal{X}. \tag{5.5}$$

For the proof of this lemma we refer to [28], [19, Lemma 5.1].

Sufficiency. Here we assume that the spectral parameter μ does not meet the condition (5.4), and establish an estimate of type (5.5) for the operator (5.3).

Introduce the Sobolev space $\mathring{H}^\ell(\mathbb{R}\times\Omega)$ of functions on the infinite cylinder $\mathbb{R}\times\Omega$ as the completion of the set $C_0^\infty(\mathbb{R}\times\Omega)$ with respect to the norm

$$\|u\|_{H^\ell(\mathbb{R}\times\Omega)} = \Big(\int_{\mathbb{R}} \sum_{r\leqslant\ell} \|\partial_x^r u\|^2_{H^{\ell-r}(\Omega)}\, dx\Big)^{1/2}, \tag{5.6}$$

where $H^{\ell-r}(\Omega)$ is the Sobolev space of functions on the compact manifold Ω. Let $L^2(\mathbb{R}\times\Omega)$ be the space induced by the product metric $dx\otimes dx + \mathfrak{h}$ on $\mathbb{R}\times\Omega$. By applying the Fourier transform $\mathcal{F}_{x\mapsto\xi}$ we pass from the continuous operator

$$\Delta_\Omega - (1+\lambda)^{-2}(\partial_x+\beta)^2 - \mu : \mathring{H}^2(\mathbb{R}\times\Omega) \to L^2(\mathbb{R}\times\Omega) \tag{5.7}$$

of the Dirichlet boundary value problem in the infinite cylinder $\mathbb{R}\times\Omega$ to the Dirichlet Laplacian $\Delta_\Omega + (1+\lambda)^{-2}(\beta+i\xi)^2 - \mu$ with the spectral parameter $\mu - (1+\lambda)^{-2}(\beta+i\xi)^2$. Assume that μ does not satisfy the equality (5.4) for any $j\in\mathbb{N}$ and $\xi\in\mathbb{R}$ or, equivalently, assume that for any $\xi\in\mathbb{R}$ the number $\mu-(1+\lambda)^{-2}(\beta+i\xi)^2$ is not an eigenvalue ν_j of the Dirichlet Laplacian Δ_Ω on $(\Omega,\mathfrak{h})$. Then a known argument, see e.g. [20, Theorem 4.1] or [18, Theorem 5.2.2] or [17, Theorem 2.4.1], shows that the operator (5.7) realizes an isomorphism. In particular, the estimate

$$\|u\|_{H^2(\mathbb{R}\times\Omega)} \leqslant C\|(\Delta_\Omega - (1+\lambda)^{-2}(\partial_x+\beta)^2 - \mu)u\|_{L^2(\mathbb{R}\times\Omega)} \tag{5.8}$$

is valid with an independent of $u\in\mathring{H}^2(\mathbb{R}\times\Omega)$ constant $C = C(\mu,\lambda,\beta) > 0$.

Let $\chi_T(x) = \chi(x-T)$, where $\chi\in C^\infty(\mathbb{R})$ is a cutoff function, such that $\chi(x) = 1$ for $x \geqslant -3$ and $\chi(x) = 0$ for $x\leqslant -4$. As a consequence of the

stabilization of the tensor field g_λ to $(1+\lambda)^2 dx\otimes dx+\mathfrak{h}$ at infinity, the operator ${}^\lambda\Delta_\beta$ stabilizes to $\Delta_\Omega-(1+\lambda)^{-2}(\partial_x+\beta)^2$ at infinity in the sense that

$$\|({}^\lambda\Delta_\beta-\Delta_\Omega+(1+\lambda)^{-2}(\partial_x+\beta)^2)\chi_T u\|_{L^2(\mathbb{R}\times\Omega)}\leqslant c(T)\|\chi_T u\|_{H^2(\mathbb{R}\times\Omega)},$$

where $c(T)\to 0$ as $T\to+\infty$, cf. (4.9) and (4.10). This together with (5.8) implies that for a sufficiently large fixed $T=T(\mu,\lambda,\beta)>0$ the estimate

$$\|\chi_T u\|_{H^2(\mathbb{R}\times\Omega)}\leqslant \mathtt{C}\|({}^\lambda\Delta_\beta-\mu)\chi_T u\|_{L^2(\mathbb{R}\times\Omega)} \tag{5.9}$$

holds, where the constant $\mathtt{C}=(1/C-c(T))^{-1}>0$ may depend on μ, λ, and β, but not on $u\in \mathring{H}^2(\mathbb{R}\times\Omega)$.

Without loss of generality we can assume that $(0,T)\times\Omega\subset\mathcal{M}_c$, cf. Fig 2. If it is not the case, then we take a larger smooth compact manifold $\mathcal{M}_c$, inserting the cylinder $(0,T)\times\Omega$ instead of the part $(0,1)\times\Omega$ of $\mathcal{M}_c$; recall that $(0,1)\times\Omega\subset\mathcal{M}_c\cap\Pi$ by our assumptions.

Let $\rho,\varsigma\in C_c^\infty(\mathcal{M})$ be some cutoff functions, such that $\rho=1$ on $\mathcal{M}\setminus(T-2,\infty)$ and $\rho=0$ on $(T-1,\infty)\times\Omega$, while $\varsigma\rho=\rho$ and $\operatorname{supp}\varsigma\subset\mathcal{M}_c$. As the operator ${}^\lambda\Delta_\beta$ is a strongly elliptic operator on $\mathcal{M}$, the local coercive estimate

$$\|\rho u\|_{H^2(\mathcal{M}_c)}\leqslant C(\|\varsigma{}^\lambda\Delta_\beta u\|+\|\varsigma u\|) \tag{5.10}$$

holds for all $u\in C_0^\infty(\mathcal{M})$. We write the estimate (5.9) for $u\in C_0^\infty(\mathcal{M})$ in the form

$$\|\chi_T u\|_{H^2(\mathbb{R}\times\Omega)}\leqslant \mathtt{C}\Big(\|\chi_T({}^\lambda\Delta_\beta-\mu)u\|_{L^2(\mathbb{R}\times\Omega)}+\|[{}^\lambda\Delta_\beta,\chi_T]u\|_{L^2(\mathbb{R}\times\Omega)}\Big),$$

where the commutator $[{}^\lambda\Delta_\beta,\chi_T]$ is equal to zero outside of the set $(T-5,T-2)\times\Omega\subset\mathcal{M}_c$. Since $\rho=1$ on this set, we get the estimate

$$\|[{}^\lambda\Delta_\beta,\chi_T]u\|_{L^2(\mathbb{R}\times\Omega)}\leqslant C\|\rho u\|_{H^2(\mathcal{M}_c)}.$$

Due to stabilization of g at infinity to the product metric $dx\otimes dx+\mathfrak{h}$ we have

$$\|\chi_T F\|^2_{L^2(\mathbb{R}\times\Omega)}=\int_{\mathbb{R}_+}\|\chi_T F\|^2_{L^2(\Omega)}\,dx\leqslant C\|F\|^2\quad\forall F\in L^2(\mathcal{M}).$$

Introduce the Sobolev space $\mathring{H}^2(\mathcal{M})$ as the completion of the set $C_0^\infty(\mathcal{M})$ with respect to the norm

$$\|u\|_{H^2(\mathcal{M})}:=\|\chi_T u\|_{H^2(\mathbb{R}\times\Omega)}+\|\rho u\|_{H^2(\mathcal{M}_c)}.$$

Then from the last four estimates it follows that

$$\|u\|_{H^2(\mathcal{M})}\leqslant C\Big(\|({}^\lambda\Delta_\beta-\mu)u\|+\|\varsigma u\|\Big), \tag{5.11}$$

where the constant C depends on λ and μ, but not on $u \in \mathring{H}^2(\mathcal{M})$. We also have the estimate

$$\|{}^{\lambda}\Delta_{\beta} u\| + \|u\| \leqslant c\|u\|_{H^2(\mathcal{M})} \quad \forall u \in \mathring{H}^2(\mathcal{M}).$$

This together with the estimate (5.11) implies that the spaces $\mathring{H}^2(\mathcal{M})$ and $\mathbf{D}({}^{\lambda}\Delta_{\beta})$ are coincident and their norms are equivalent.

Let w be a bounded rapidly decreasing at infinity positive function on $\mathcal{M}$, such that the embedding of $\mathring{H}^2(\mathcal{M})$ into the weighted space $L^2(\mathcal{M}, \mathsf{w})$ with the norm $\|\mathsf{w}\cdot\|$ is compact. As a consequence of (5.11) we obtain the estimate

$$\|u\|_{H^2(\mathcal{M})} \leqslant \mathrm{C}\Big(\|({}^{\lambda}\Delta_{\beta} - \mu)u\| + \|\mathsf{w}u\|\Big) \quad \forall u \in \mathring{H}^2(\mathcal{M}) \tag{5.12}$$

of type (5.5). Then by the Peetre's lemma the range of the continuous operator ${}^{\lambda}\Delta_{\beta} - \mu : \mathring{H}^2(\mathcal{M}) \to L^2(\mathcal{M})$ is closed and the kernel is finite dimensional. In order to see that the cokernel of this operator is finite dimensional, one can apply a similar argument to the adjoint operator ${}^{\lambda}\Delta_{\beta}^{*} = e^{\overline{\beta}\mathsf{s}} \frac{1}{\varrho_{\overline{\lambda}}} {}^{\overline{\lambda}}\Delta \varrho_{\overline{\lambda}} e^{-\overline{\beta}\mathsf{s}}$. The operator ${}^{\lambda}\Delta_{\beta}^{*}$ stabilizes at infinity to the operator $\Delta_{\Omega} - (1 + \overline{\lambda})^{-2}(\overline{\beta} - \partial_x)^2$. If μ does not meet the condition (5.4), this allows to deduce the estimate

$$\|u\|_{H^2(\mathcal{M})} \leqslant \mathrm{C}\Big(\|({}^{\lambda}\Delta_{\beta} - \mu)^{*}u\| + \|\mathsf{w}u\|\Big) \quad \forall u \in \mathring{H}^2(\mathcal{M}),$$

which implies that the cokernel $\operatorname{coker}({}^{\lambda}\Delta - \mu) = \ker({}^{\lambda}\Delta - \mu)^{*}$ is finite dimensional. Thus the deformed Dirichlet Laplacian ${}^{\lambda}\Delta_{\beta} - \mu : \mathbf{D}({}^{\lambda}\Delta_{\beta}) \to L^2(\mathcal{M})$ is Fredholm, if μ does not meet the condition (5.4).

Necessity. Now we assume that μ meets the condition (5.4) for some j, and show that the operator ${}^{\lambda}\Delta_{\beta} - \mu : \mathring{H}^2(\mathcal{M}) \to L^2(\mathcal{M})$ is not Fredholm. By the Peetre's lemma it suffices to find a sequence $\{u_{\ell}\}_{\ell=1}^{\infty}$ of functions $u_{\ell} \in \mathring{H}^2(\mathcal{M})$ violating the estimate (5.12).

Let χ be a smooth cutoff function on the real line, such that $\chi(x) = 1$ for $|x - 3| \leqslant 1$, and $\chi(x) = 0$ for $|x - 3| \geqslant 2$. Consider the functions

$$u_{\ell}(x, \mathrm{y}) = \chi(x/\ell) \exp\Big(i(1+\lambda)x\sqrt{\mu - \nu_j} - \beta x\Big)\Phi(\mathrm{y}), \quad (x, \mathrm{y}) \in \mathbb{R} \times \Omega, \tag{5.13}$$

where Φ is an eigenfunction of the Dirichlet Laplacian Δ_{Ω} corresponding to the eigenvalue ν_j. It is clear that u_{ℓ} satisfies the homogeneous Dirichlet boundary condition on $\mathbb{R} \times \partial\Omega$. As μ meets the condition (5.4), the exponent in (5.13) is an oscillating function of $x \in \mathbb{R}$. Straightforward calculation shows that

$$\Big\|\Big(\Delta_{\Omega} - (1+\lambda)^{-2}(\partial_x + \beta)^2 - \mu\Big)u_{\ell}\Big\|_{L^2(\mathbb{R}\times\Omega)} \leqslant \mathsf{c}, \quad \|u_{\ell}\|_{H^2(\mathbb{R}\times\Omega)} \to \infty \tag{5.14}$$

as $\ell \to +\infty$, where the constant c is independent of ℓ. We extend the functions u_ℓ from their supports in Π to $\mathcal{M}$ by zero.

Assume that the estimate (5.12) is valid. Without loss of generality we can take a rapidly decreasing weight w, such that the embedding $\overset{\circ}{H}{}^2(\mathcal{M}) \hookrightarrow L^2(\mathcal{M}; \mathsf{w})$ is compact, and $\|\mathsf{w}u_\ell\| \leqslant \mathsf{C}$ uniformly in $\ell \geqslant 1$. Due to stabilization of ${}^\lambda\Delta_\beta$ to $\Delta_\Omega - (1+\lambda)^{-2}(\partial_x + \beta)^2$ at infinity we have

$$\|({}^\lambda\Delta - \Delta_\Omega + (1+\lambda)^{-2}(\partial_x+\beta)^2)u_\ell\| \leqslant \mathsf{c}_\ell \|u_\ell\|_{H^2(\mathbb{R}\times\Omega)},$$

where $\mathsf{c}_\ell \to 0$ as $\ell \to +\infty$. This together with (5.14) gives

$$\begin{aligned} \|({}^\lambda\Delta_\beta - \mu)u_\ell\| \leqslant C\Big\|\Big(-(1+\lambda)^{-2}(\partial_x+\beta)^2 + \Delta_\Omega - \mu\Big)u_\ell\Big\|_{L^2(\mathbb{R}\times\Omega)} \\ +\|({}^\lambda\Delta - \Delta_\Omega + (1+\lambda)^{-2}(\partial_x+\beta)^2)u_\ell\| \leqslant C\mathsf{c} + \mathsf{c}_\ell\|u_\ell\|_{H^2(\mathbb{R}\times\Omega)}. \end{aligned} \tag{5.15}$$

Finally, as a consequence of (5.12) and (5.15) we get

$$\begin{aligned} \|u_\ell\|_{H^2(\mathbb{R}\times\Omega)} &\leqslant \mathrm{C}\Big(\|({}^\lambda\Delta - \mu)u_\ell\| + \|\mathsf{w}u_\ell\|\Big) \\ &\leqslant \mathrm{C}(C\mathsf{c} + \mathsf{c}_\ell\|u_\ell\|_{H^2(\mathbb{R}\times\Omega)} + \mathsf{C}). \end{aligned} \tag{5.16}$$

Since $\mathsf{c}_\ell \to 0$, the inequalities (5.16) imply that the value $\|u_\ell\|_{H^2(\mathbb{R}\times\Omega)}$ remains bounded as $\ell \to +\infty$. This contradicts (5.14). Thus the sequence $\{u_\ell\}_{\ell=1}^\infty$ violates the estimate (5.12). The necessity is proven. □

Corollary 5.4 *All eigenvalues of the Laplacian Δ on a manifold with an asymptotically cylindrical end are of finite multiplicity.*

PROOF. By Proposition 5.3 for every μ there exists $\beta > 0$, such that the operator ${}^0\Delta_\beta - \mu : \mathbf{D}({}^0\Delta_\beta) \to L^2(\mathcal{M})$ is Fredholm. From $\Psi \in \ker(\Delta - \mu)$ it follows that $e^{-\beta\mathsf{s}}\Psi \in \ker({}^0\Delta_\beta - \mu)$. As a consequence,

$$\dim\ker(\Delta - \mu) \leqslant \dim\ker({}^0\Delta_\beta - \mu) < \infty.$$

□

6 Exponential decay of the non-threshold eigenfunctions

Observe that the set $\mathbb{C} \setminus \sigma_{ess}({}^\lambda\Delta)$ is simply connected. A standard argument based on the analytic Fredholm theory shows that the spectrum $\sigma({}^\lambda\Delta)$ of the m-sectorial operator ${}^\lambda\Delta$ is the union of the essential spectrum $\sigma_{ess}({}^\lambda\Delta)$ and the discrete spectrum $\sigma_d({}^\lambda\Delta)$, e.g. [12,30], [17, Appendix].

As in the theory of N-body Schrödinger operators, see e.g. [30, Chapter XII.11] and references therein, it is much easier to prove exponential decay of the eigenfunctions corresponding to the isolated eigenvalues. In our case this does not require any assumptions on the analytic regularity of the metric g, and we can consider a general manifold with an asymptotically cylindrical end in the sense of Definition 2.2. However, only the Dirichlet Laplacian may have isolated eigenvalues below the first threshold $\nu_1 > 0$. For the Neumann Laplacian and for the Laplacian on a manifold without boundary the first threshold ν_1 is zero and the eigenvalues are embedded into the essential spectrum $\sigma_{ess}(\Delta) = [0, \infty)$.

In the next lemma we study the eigenfunctions of $\sigma_d({}^\lambda\Delta)$. In particular, this lemma implies exponential decay of the eigenfunctions corresponding to the isolated eigenvalues of the Dirichlet Laplacian.

Lemma 6.1 *Assume that $(\mathcal{M}, \mathsf{g})$ is a manifold with an axial analytic asymptotically cylindrical end, and $\lambda \in \mathcal{D}_\alpha$ is fixed. Then for $\mu \in \sigma_d({}^\lambda\Delta)$ and $\Psi \in \ker({}^\lambda\Delta - \mu)$ we have $e^{-\beta\mathsf{s}}\Psi \in \ker({}^\lambda\Delta_\beta - \mu) \subset L^2(\mathcal{M})$ with some $\beta < 0$. In other words, the eigenfunctions corresponding to an isolated eigenvalue of ${}^\lambda\Delta$ are exponentially decaying at infinity in the mean.*

For a general manifold $(\mathcal{M}, \mathsf{g})$ with an asymptotically cylindrical end the assertion remains valid for $\lambda = 0$.

PROOF. Recall from [16,30] that the family of m-sectorial operators $\mathbb{C} \ni \beta \mapsto {}^\lambda\Delta_\beta$ is said to be analytic of type (B), if for any $\beta \in \mathbb{C}$ the sectorial form q_λ^β is densely defined and closed, its domain $\mathbf{D}(\mathsf{q})$ is independent of β, and the function $\mathbb{C} \ni \beta \mapsto \mathsf{q}_\lambda^\beta[u, u]$ is analytic for any $u \in \mathbf{D}(\mathsf{q})$. Thus the family $\mathbb{C} \ni \beta \mapsto {}^\lambda\Delta_\beta$ is analytic of type (B) by Proposition 5.2. As is known [16,30], this implies that for some $\epsilon > 0$ there exist functions $\mu_1, \dots, \mu_k$ in the disk $\{\beta \in \mathbb{C} : |\beta| < \epsilon\}$ with at worst algebraic branching point at $\beta = 0$, such that the spectrum of ${}^\lambda\Delta_\beta$ in a small neighborhood $\mathcal{O}$ of $\mu \in \sigma_d({}^\lambda\Delta_0)$ consists of the isolated eigenvalues $\mu_1(\beta), \dots, \mu_k(\beta)$. Moreover, in the same disk $|\beta| < \epsilon$ there is the analytic function

$$\beta \mapsto \mathsf{P}(\beta) = \frac{1}{2\pi i} \oint_{\partial\mathcal{O}} ({}^\lambda\Delta_\beta - \zeta)^{-1} \, d\zeta,$$

whose values are the projections onto the generalized eigenspace of ${}^\lambda\Delta_\beta$ associated with the eigenvalues $\mu_1(\beta), \dots, \mu_k(\beta)$. For all $\beta \in i\mathbb{R}$ and $u \in \mathbf{D}({}^\lambda\Delta)$ we have $e^{\beta\mathsf{s}}\, {}^\lambda\Delta_\beta \, e^{-\beta\mathsf{s}} u = {}^\lambda\Delta u$. Therefore $\mu_j(\beta) = \dots = \mu_k(\beta) = \mu$ and

$$\mathsf{P}(\beta) e^{-\beta\mathsf{s}} f = e^{-\beta\mathsf{s}} \mathsf{P}(0) f, \quad f \in C_0^\infty(\mathcal{M}), \tag{6.1}$$

for all $\beta \in i\mathbb{R}$. By analyticity these equalities extend to the disk $|\beta| < \epsilon$. The set $C_0^\infty(\mathcal{M})$ is dense in $L^2(\mathcal{M})$, and the range of $\mathsf{P}(0)$ is finite dimensional.

Hence for any $\Psi \in \ker({}^\lambda\Delta - \mu)$ we have $\Psi = \mathsf{P}(0)f$ with some $f \in C_0^\infty(\mathcal{M})$. Due to (6.1) the function $i\mathbb{R} \ni \beta \mapsto e^{-\beta \mathsf{s}}\Psi \in L^2(\mathcal{M})$ extends by analyticity to the disk $|\beta| < \epsilon$. Clearly, $e^{-\beta\mathsf{s}}\Psi \in \ker({}^\lambda\Delta_\beta - \mu)$.

It can be shown that the equality (6.1) and the inclusions $\mu \in \sigma_d({}^\lambda\Delta_\beta)$, $e^{-\beta\mathsf{s}}\Psi \in \ker({}^\lambda\Delta_\beta - \mu)$ remain valid as β varies in a neighborhood of zero so that the parabolas of $\sigma_{ess}({}^\lambda\Delta_\beta)$ do not cover the point μ, cf. Proposition 5.3 and Fig. 4. Let us also note that an independent proof of the assertion can be obtained by methods of the asymptotic theory in [20], see also [18, Chapter 5]. $\square$

As is well-known e.g. [30, Chapter XII.11], for N-body Schrödinger operators with dilation analytic potentials it is also possible to prove exponential decay of all non-threshold eigenfunctions. In the next theorem we show that a similar argument, based on the complex scaling and the Phragmén-Lindelöf principle, allows to prove exponential decay of the non-threshold eigenfunctions of the Laplacian on a manifold with an axial analytic asymptotically cylindrical end.

Theorem 6.2 *Let Ψ be a non-threshold eigenfunction of the Laplacian Δ on a manifold with an axial analytic asymptotically cylindrical end; i.e. $\Delta\Psi = \mu\Psi \in L^2(\mathcal{M})$ with $\mu \in \sigma(\Delta) \setminus \{\nu_j\}_{j=1}^\infty$. Then the estimate* (3.1) *holds for some $\gamma < 0$ and an independent of x constant C.*

The proof of theorem is preceded by the following lemma.

Lemma 6.3 *Let the assumptions of Theorem 6.2 be fulfilled. Then the function $\lambda \mapsto \Psi \circ \varkappa_\lambda \in L^2(\mathcal{M})$ extends by analyticity from real to all λ in the disk $\mathcal{D}_\alpha$. Moreover, $\Psi \circ \varkappa_\lambda \in \ker({}^\lambda\Delta - \mu)$ for all $\lambda \in \mathcal{D}_\alpha$, and $\mu \in \sigma_d({}^\lambda\Delta)$ for all non-real $\lambda \in \mathcal{D}_\alpha$. (Recall that $\varkappa_\lambda$ with $\lambda \in \mathbb{R} \cap \mathcal{D}_\alpha$ is the selfdiffeomorphism of $\mathcal{M}$ defined in Section 4, and $\alpha < \pi/4$ is an angle for which the conditions of Definition 2.1 are fulfilled.)*

PROOF of Lemma 6.3 As preliminaries to the proof we briefly recall a construction from [12].

For $\lambda \in \mathcal{D}_\alpha \cap \mathbb{R}$ the operator ${}^\lambda\Delta$ is the Laplacian on $(\mathcal{M}, \varkappa_\lambda^*\mathsf{g})$, and the Riemannian geometry gives the identity

$$(\Delta - \zeta)u = \big(({}^\lambda\Delta - \zeta)(u \circ \varkappa_\lambda)\big) \circ \varkappa_\lambda^{-1} \quad \forall u \in \mathbf{C}({}^0\Delta), \tag{6.2}$$

where $u \circ \varkappa_\lambda$ is in the core $\mathbf{C}({}^\lambda\Delta)$ introduced in Definition 4.1. Let $\zeta < 0$ be outside of the sector of the m-sectorial operators ${}^\lambda\Delta$, $\lambda \in \mathcal{D}_\alpha$. Then the resolvent $({}^\lambda\Delta - \zeta)^{-1}$ is an analytic function of $\lambda \in \mathcal{D}_\alpha$, and we can rewrite (6.2)

in the form

$$(\Delta-\zeta)^{-1}F=\Big(({}^\lambda\Delta-\zeta)^{-1}(F\circ\varkappa_\lambda)\Big)\circ\varkappa_\lambda^{-1},\quad \lambda\in\mathcal{D}_\alpha\cap\mathbb{R}, \tag{6.3}$$

where F is in the dense in $L^2(\mathcal{M})$ subset $\{F=(\Delta-\zeta)u : u\in \mathbf{C}({}^0\Delta)\}$. The equality (6.3) extends by continuity to all $F\in L^2(\mathcal{M})$. Taking the inner product of (6.3) with $G\in L^2(\mathcal{M})$, we obtain the identity

$$\Big((\Delta-\zeta)^{-1}F,G\Big)=\Big(({}^\lambda\Delta-\zeta)^{-1}(F\circ\varkappa_\lambda),G\circ\varkappa_{\overline{\lambda}}\Big)_\lambda \tag{6.4}$$

for all $\lambda\in\mathcal{D}_\alpha\cap\mathbb{R}$. The identity (6.4) cannot be extended by analyticity to all $\lambda\in\mathcal{D}_\alpha$ for arbitrary $F,G\in L^2(\mathcal{M})$. However it can be done for all F and G in some subset $\mathcal{A}\subset L^2(\mathcal{M})$ of analytic vectors.

In order to introduce the set $\mathcal{A}$, consider the algebra $\mathscr{E}$ of all entire functions $\mathbb{C}\ni z\mapsto f(z,\cdot)\in C^\infty(\Omega)$, such that in any sector $|\Im z|\leqslant(1-\epsilon)\Re z$ with $\epsilon>0$ the value $\|f(z,\cdot)\|_{L^2(\Omega)}$ decays faster than any inverse power of $\Re z$ as $\Re z\to+\infty$. By definition a function $F\in L^2(\mathcal{M})$ is in the subset $\mathcal{A}$ of analytic vectors, if $F(x,\mathrm{y})=f(x,\mathrm{y})$ for some $f\in\mathscr{E}$ and all $(x,\mathrm{y})\in\Pi$. For $F\in\mathcal{A}$ and $\lambda\in\mathbb{C}$ we define the function $F\circ\varkappa_\lambda$ on $\mathcal{M}$, such that $F\circ\varkappa_\lambda\equiv F$ on $\mathcal{M}\setminus\Pi$, and

$$F\circ\varkappa_\lambda(x,\mathrm{y})=f(x+\lambda\mathsf{s}_R(x),\mathrm{y}),\quad (x,\mathrm{y})\in\Pi. \tag{6.5}$$

Here $f(x+\lambda\mathsf{s}_R(x),\cdot)$ is the value of the corresponding to F entire function $f\in\mathscr{E}$ at the point $z=x+\lambda\mathsf{s}_R(x)$, and $\varkappa_\lambda(x,\mathrm{y})=(x+\lambda\mathsf{s}_R(x),\mathrm{y})$ is the complex scaling in Π. By [12, Lemma 7.1] we have:

i. For any $F\in\mathcal{A}$, $\mathcal{D}_\alpha\ni\lambda\mapsto F\circ\varkappa_\lambda$ is an $L^2(\mathcal{M})$-valued analytic function;
ii. For any $\lambda\in\mathcal{D}_\alpha$ the image $\varkappa_\lambda[\mathcal{A}]=\{F\circ\varkappa_\lambda : F\in\mathcal{A}\}$ of $\mathcal{A}$ under $\varkappa_\lambda$ is dense in the space $L^2(\mathcal{M})$.

For $F,G\in\mathcal{A}$ the equality (6.4) extends by analyticity to all $\lambda\in\mathcal{D}_\alpha$. Let μ be the same as in the assertion of the lemma. Then $\mu\notin\sigma_{ess}({}^\lambda\Delta)$ for all non-real $\lambda\in\mathcal{D}_\alpha$ by Proposition 5.3. By applying the Aguilar-Balslev-Combes argument to the equality (6.4), one can see that for any $\lambda\in\mathcal{D}_\alpha\setminus\mathbb{R}$ the resolvent $({}^\lambda\Delta-\zeta)^{-1}$ is an analytic function of ζ in a small complex neighborhood of μ, except for the point $\mu\in\sigma_d({}^\lambda\Delta)\cap\mathbb{R}$ itself, where the resolvent $({}^\lambda\Delta-\zeta)^{-1}$ has a simple pole; for details we refer to [12]. Now the preliminaries are complete, and we are in position to prove the assertion.

Let $\eta\in\mathcal{D}_\alpha\cap\mathbb{R}$. Then the Laplacian ${}^\eta\Delta$ and the projection

$$\mathsf{P}(\eta)=\text{s-lim}_{\epsilon\downarrow 0}\, i\epsilon({}^\eta\Delta-\mu+i\epsilon)^{-1}$$

onto its eigenspace corresponding to the eigenvalue μ are selfadjoint with respect to the inner product $(\cdot,\cdot)_\eta$ in $L^2(\mathcal{M})$. If $\lambda\in\mathcal{D}_\alpha\setminus\mathbb{R}$, then we define the projection $\mathsf{P}(\lambda)$ onto the eigenspace of the non-selfadjoint operator ${}^\lambda\Delta$

associated with the eigenvalue $\mu \in \sigma_d({}^\lambda\Delta) \cap \mathbb{R}$ as the Riesz projection; i.e. as the first order residue of the resolvent $({}^\lambda\Delta - \zeta)^{-1}$ at the simple pole μ. The resolvent $({}^\lambda\Delta - \zeta)^{-1}$ is an analytic function of two variables on the set $\{(\lambda, \zeta) : \lambda \in \mathcal{D}_\alpha, \mu \in \mathbb{C} \setminus \sigma({}^\lambda\Delta)\}$, e.g. [30, Theorem XII.7]. Therefore the Riesz projection $\mathsf{P}(\lambda)$ is an analytic function of λ on the set $\mathcal{D}_\alpha \setminus \mathbb{R}$.

As a consequence of the equality (6.4), for all $F, G \in \mathcal{A}$ we get

$$\Big(\mathsf{P}(0)F, G\Big) = \Big(\mathsf{P}(\eta)(F \circ \varkappa_\eta), G \circ \varkappa_\eta\Big)_\eta = \Big(\mathsf{P}(\lambda)(F \circ \varkappa_\lambda), G \circ \varkappa_{\overline{\lambda}}\Big)_\lambda. \tag{6.6}$$

Recall that the equality $(\varrho_\lambda \mathcal{F}, \mathcal{G})_\lambda = (\mathcal{F}, \mathcal{G})$ is valid for all $\mathcal{F}, \mathcal{G} \in L^2(\mathcal{M})$, where $\mathcal{D}_\alpha \ni \lambda \mapsto \varrho_\lambda \in C^\infty(\mathcal{M})$ is an analytic function satisfying (4.7). Since the sets $\varkappa_\lambda[\mathcal{A}]$ and $\varkappa_{\overline{\lambda}}[\mathcal{A}]$ are dense in $L^2(\mathcal{M})$, by the equalities (6.6) we have

$$\begin{aligned}
\|\mathsf{P}(\lambda)\| &= \sup_{F,G\in\mathcal{A}} \frac{\Big(\mathsf{P}(\eta)(F \circ \varkappa_\eta), G \circ \varkappa_\eta\Big)_\eta}{\|F \circ \varkappa_\lambda/\sqrt{\varrho_\lambda}\| \|G \circ \varkappa_{\overline{\lambda}}/\sqrt{\varrho_{\overline{\lambda}}}\|} \\
&\leqslant \sup_{F,G\in\mathcal{A}} \frac{\|F \circ \varkappa_\eta/\sqrt{\varrho_\eta}\| \|G \circ \varkappa_\eta/\sqrt{\varrho_\eta}\|}{\|F \circ \varkappa_\lambda/\sqrt{\varrho_\lambda}\| \|G \circ \varkappa_{\overline{\lambda}}/\sqrt{\varrho_{\overline{\lambda}}}\|} \to 1 \text{ as } \lambda \to \eta, \lambda \in \mathcal{D}_\alpha \setminus \mathbb{R}.
\end{aligned}$$

Thanks to (6.6) we also have

$$\begin{aligned}
&\Big(\mathsf{P}(\lambda)(F \circ \varkappa_\lambda) - \mathsf{P}(\eta)(F \circ \varkappa_\eta), G \circ \varkappa_\eta\Big)_\eta \\
=&\Big(\mathsf{P}(\lambda)(F \circ \varkappa_\lambda), G \circ \varkappa_\eta/\varrho_\eta - G \circ \varkappa_{\overline{\lambda}}/\varrho_{\overline{\lambda}}\Big) \to 0 \text{ as } \lambda \to \eta.
\end{aligned} \tag{6.7}$$

Here the right hand side tends to zero because the norm $\|\mathsf{P}(\lambda)(F \circ \varkappa_\lambda)\|$ remains bounded, while $G \circ \varkappa_{\overline{\lambda}}/\varrho_{\overline{\lambda}}$ tends to $G \circ \varkappa_\eta/\varrho_\eta$ in $L^2(\mathcal{M})$ as $\lambda \to \eta$. The set $\{G \circ \varkappa_\eta : G \in \mathcal{A}\}$ is dense in $L^2(\mathcal{M})$, and hence (6.7) implies that $\mathsf{P}(\lambda)(F \circ \varkappa_\lambda)$ weakly converges to $\mathsf{P}(\eta)(F \circ \varkappa_\eta)$ as $\lambda \to \eta$, e.g. [16, Lemma III.1.31]. Therefore the function $\lambda \mapsto \mathsf{P}(\lambda)(F \circ \varkappa_\lambda) \in L^2(\mathcal{M})$ is weakly (and therefore strongly) analytic in the whole disk $\mathcal{D}_\alpha$ for any $F \in \mathcal{A}$. Due to the equality (6.3) we have $\Big(\mathsf{P}(0)F\Big) \circ \varkappa_\lambda = \mathsf{P}(\lambda)(F \circ \varkappa_\lambda)$ (first for all real, and then by analyticity) for all $\lambda \in \mathcal{D}_\alpha$. Since the range $\operatorname{Ran} \mathsf{P}(0) = \ker(\Delta - \mu)$ is finite dimensional and the set $\mathcal{A}$ is dense in $L^2(\mathcal{M})$, for any $\Psi \in \ker(\Delta - \mu)$ there exists $F \in \mathcal{A}$, such that $\Psi = \mathsf{P}(0)F$. The right hand side of the equality $\Psi \circ \varkappa_\lambda = \mathsf{P}(\lambda)(F \circ \varkappa_\lambda)$ provides the left hand side with an analytic continuation in $\lambda \in \mathcal{D}_\alpha$. The continuation takes its values in the space $L^2(\mathcal{M})$. It remains to note that $\operatorname{Ran} \mathsf{P}(\lambda) = \ker({}^\lambda\Delta - \mu)$ as $\mu \in \sigma_d({}^\lambda\Delta) \cap \mathbb{R}$ is a simple pole of the resolvent $({}^\lambda\Delta - \zeta)^{-1}$, e.g. [16]. $\square$

PROOF of Theorem 6.2 In the first part of the proof we establish the analyticity of Ψ with respect to x in a complex conical neighborhood of infinity. Then (3.1) follows from a variant of the the Phragmén-Lindelöf principle.

For brevity we denote $\Psi_\lambda = \Psi \circ \varkappa_\lambda$. By Lemma 6.3 the function $\mathcal{D}_\alpha \ni \lambda \mapsto \Psi_\lambda \in L^2(\mathcal{M})$ is analytic, and $\Psi_\lambda \in \ker({}^\lambda\Delta - \mu)$. (Note that $\ker({}^\lambda\Delta - \mu) \subset C^\infty(\mathcal{M})$ by usual results on local regularity of solutions to elliptic problems.) We have

$$\mathsf{q}_\lambda[\Psi_\lambda, v] - \mu(\Psi_\lambda, v)_\lambda = 0 \quad \forall v \in \mathbf{D}(\mathsf{q}).$$

This together with Proposition 4.3.ii and (4.8) gives

$$\|\Psi_\lambda\|^2_{\mathbf{D}(\mathsf{q})} = (d\Psi_\lambda, d\Psi_\lambda) + \|\Psi_\lambda\|^2 \leqslant (b^{-1}(c\mu + a) + 1)\|\Psi_\lambda\|^2.$$

As is known [16, Chapter VII.4], results of Proposition 4.3 also imply that

$$|\mathsf{q}_\lambda[u, v] - \mathsf{q}_\varsigma[u, v]| \leqslant C_{\lambda,\varsigma}\|u\|_{\mathbf{D}(\mathsf{q})}\|v\|_{\mathbf{D}(\mathsf{q})},$$

where the constant $C_{\lambda,\varsigma}$ is bounded uniformly in $u, v \in \mathbf{D}(\mathsf{q})$ and $\lambda, \varsigma \in \mathcal{D}_\alpha$; moreover, $C_{\lambda,\varsigma} \to 0$ as $|\lambda - \varsigma| \to 0$. As a consequence we obtain

$$\begin{aligned} b\|\Psi_\lambda - \Psi_\varsigma\|^2_{\mathbf{D}(\mathsf{q})} - (a + b + c\mu)\|\Psi_\lambda - \Psi_\varsigma\|^2 \\ \leqslant \Re\mathsf{q}_\lambda[\Psi_\lambda - \Psi_\varsigma, \Psi_\lambda - \Psi_\varsigma] - \mu\Re(\Psi_\lambda - \Psi_\varsigma, \Psi_\lambda - \Psi_\varsigma)_\lambda \\ \leqslant \Big|\mathsf{q}_\lambda[\Psi_\varsigma, \Psi_\lambda - \Psi_\varsigma] - \mu(\Psi_\varsigma, \Psi_\lambda - \Psi_\varsigma)_\lambda - \mathsf{q}_\varsigma[\Psi_\varsigma, \Psi_\lambda - \Psi_\varsigma] + \mu(\Psi_\varsigma, \Psi_\lambda - \Psi_\varsigma)_\varsigma\Big| \\ \leqslant \Big|\mathsf{q}_\lambda[\Psi_\varsigma, \Psi_\lambda - \Psi_\varsigma] - \mathsf{q}_\varsigma[\Psi_\varsigma, \Psi_\lambda - \Psi_\varsigma]\Big| + \mu\Big|\Big((\varrho_\lambda - \varrho_\varsigma)\Psi_\varsigma, \Psi_\lambda - \Psi_\varsigma\Big)\Big| \\ \leqslant \mathrm{C}_{\lambda,\varsigma}\|\Psi_\varsigma\|\|\Psi_\lambda - \Psi_\varsigma\|_{\mathbf{D}(\mathsf{q})}, \end{aligned}$$

where $\mathrm{C}_{\lambda,\varsigma}$ is uniformly bounded, and $\mathrm{C}_{\lambda,\varsigma} \to 0$ as $|\lambda - \varsigma| \to 0$. By these estimates the analyticity of $\mathcal{D}_\alpha \ni \lambda \mapsto \Psi_\lambda \in L^2(\mathcal{M})$ leads to the continuity of the function $\mathcal{D}_\alpha \ni \lambda \mapsto \Psi_\lambda \in \mathbf{D}(\mathsf{q})$. Then by the Morera's theorem the function $\mathcal{D}_\alpha \ni \lambda \mapsto \Psi_\lambda \in \mathbf{D}(\mathsf{q})$ is analytic. The graph norm of $\Delta^{1/2}$ is an equivalent norm in $\mathbf{D}(\mathsf{q})$ e.g. [16], and therefore usual results on traces of functions in the Sobolev space $H^1_{loc}(\mathcal{M}) \supseteq \mathbf{D}(\mathsf{q})$ apply. In particular, for any fixed $x \in \mathbb{R}_+$ we have

$$\|u(x)\|_{L^2(\Omega)} \leqslant \|u(x)\|_{H^{1/2}(\Omega)} \leqslant c\|u\|_{\mathbf{D}(\mathsf{q})},$$

where c is independent of $u \in \mathbf{D}(\mathsf{q})$. Hence for any $x \in \mathbb{R}_+$ the function $\mathcal{D}_\alpha \ni \lambda \mapsto \Psi_\lambda(x) \in L^2(\Omega)$ is analytic.

Consider $\Psi_\lambda(x) = \Psi(x + \lambda\mathsf{s}_R(x))$ as an analytic $L^2(\Omega)$-valued function of $z = x + \lambda\mathsf{s}_R(x)$ in the complex conical neighborhood

$$\mathcal{S}_\epsilon = \{x + \lambda\mathsf{s}_R(x) \in \mathbb{C} : x \in \mathbb{R}_+, \mathsf{s}_R(x) \neq 0, |\lambda| \leqslant \epsilon\}, \quad \epsilon < \sin\alpha,$$

of infinity. The function Ψ is uniformly bounded on $\mathcal{S}_\epsilon$ in the sense that

$$\int_{\mathfrak{L}^R_\lambda} \|\Psi(z)\|^2_{L^2(\Omega)}\,|dz| = \int_0^\infty \|\Psi_\lambda(x)\|^2_{L^2(\Omega)}|1 + \lambda\mathsf{s}'_R(x)|\,dx \leqslant 2\|\Psi_\lambda\|^2 \leqslant C, \quad (6.8)$$

where $\mathfrak{L}^R_\lambda$ is the curve depicted on Fig. 3, and $|\lambda| \leqslant \epsilon$. By Lemmas 6.3 and 6.1 for any non-real $\lambda \in \mathcal{D}_\alpha$ we have $\mu \in \sigma_d({}^\lambda\Delta)$ and $e^{-\beta(\lambda)\mathsf{s}}\Psi_\lambda \in L^2(\mathcal{M})$ with some

negative $\beta(\lambda)$. Therefore the function $\mathcal{S}_\epsilon \ni z \mapsto \Psi(z) \in L^2(\Omega)$ is exponentially decaying outside of the half-axis $\mathbb{R}_+$ in the sense that

$$\int_{\mathfrak{L}^R_\lambda} \|e^{-\beta(\lambda)z/2}\Psi(z)\|^2_{L^2(\Omega)}\,|dz| \leqslant C(\lambda) < \infty, \quad \beta(\lambda) < 0, |\lambda| \leqslant \epsilon, \lambda \notin \mathbb{R}. \quad (6.9)$$

In the remaining part of the proof we derive a variant of the Phragmén-Lindelöf principle, which says that any analytic function $\mathcal{S}_\epsilon \ni z \mapsto \Psi(z) \in L^2(\Omega)$ satisfying (6.8) and (6.9) also meets the estimate (3.1) with some $\gamma < 0$.

The uniform estimate (6.8) necessitates existence of a sequence of positive numbers $\{T_\ell\}_{\ell=1}^\infty$, $T_\ell \to \infty$ as $\ell \to \infty$, such that for a given $\gamma < 0$ and any $\delta > 0$ the integral

$$\int_{\{z\in\mathcal{S}_\epsilon: z = T_\ell + \lambda \mathsf{s}_R(T_\ell), |\lambda|\leqslant\epsilon, |1+\lambda|=1\}} \frac{e^{-\delta z^2 - \gamma z}\Psi(z)}{x - z}\,dz, \quad x \in \mathbb{R}_+,$$

tends to zero in $L^2(\Omega)$ as $\ell \to \infty$. As a consequence, the contour of the Cauchy integral in the equality

$$e^{-\delta x^2 - \gamma x}\Psi(x) = \frac{1}{2\pi i}\oint_{|x-z|=\wp(x)} \frac{e^{-\delta z^2-\gamma z}\Psi(z)}{x-z}\,dz, \quad x \in \mathcal{S}_\epsilon \cap \mathbb{R}_+,$$

(where $\wp(x)$ is so small that the contour of integration lies in $\mathcal{S}_\epsilon$) can be deformed so that

$$e^{-\delta x^2-\gamma x}\Psi(x) = \frac{1}{2\pi i}\Big(\int_{\mathfrak{L}^R_{i\epsilon}} - \int_{\mathfrak{L}^R_{-i\epsilon}}\Big)\frac{e^{-\delta z^2-\gamma z}\Psi(z)}{x-z}\,dz. \quad (6.10)$$

Here both integrals are absolutely convergent in $L^2(\Omega)$ because of (6.8). Moreover, thanks to (6.9), for $0 > \gamma > \max\{\beta(i\epsilon), \beta(-i\epsilon)\}/2$ these integrals are bounded in $L^2(\Omega)$ uniformly in $\delta > 0$ and large $x > 0$. Hence the same is true for the left hand side of the equality (6.10). This immediately leads to (3.1). $\square$

7 Refined exponential decay of the non-threshold eigenfunctions and accumulation of eigenvalues

In this section we consider the Laplacian on a general manifold $(\mathcal{M}, \mathsf{g})$ with an asymptotically cylindrical end $(\Pi, \mathsf{g}\restriction_\Pi)$ in the sense of Definition 2.2. The axial analyticity of the end is not assumed. We study the non-threshold exponentially decaying eigenfunctions and accumulation of the corresponding eigenvalues. Let us stress that in the case of an axial analytic asymptotically cylindrical end $(\Pi, \mathsf{g}\restriction_\Pi)$ all non-threshold eigenfunctions of the Laplacian are of some exponential decay by Theorem 6.2.

Theorem 7.1 (Refined exponential decay) *Let Ψ be a non-threshold exponentially decaying eigenfunction of the Laplacian on a manifold $(\mathcal{M}, \mathsf{g})$ with an asymptotically cylindrical end; i.e. $\Delta\Psi = \mu\Psi$ with $\mu \in \sigma(\Delta) \setminus \{\nu_j\}_{j=1}^{\infty}$, and the estimate* (3.1) *holds for some $\gamma < 0$. Then the estimate* (3.1) *holds for any negative $\gamma > -\min_{j:\nu_j>\mu} \sqrt{\nu_j - \mu}$.*

PROOF. In the proof we use methods of the asymptotic theory [17,18,20].

Let us consider the case of the Dirichlet Laplacian on a manifold $(\mathcal{M}, \mathsf{g})$ with non-compact boundary. The case $\partial\mathcal{M} = \varnothing$ is similar, while in the case of the Neumann Laplacian some changes in our argument are needed, which are outlined in the end of the proof.

Take a cutoff function $\chi \in C^\infty(\mathbb{R})$, such that $0 \leqslant \chi(x) \leqslant 1$, $\chi(x) = 0$ for $x \leqslant 1$, and $\chi(x) = 1$ for $x \geqslant 2$. Consider the infinite cylinder $\mathbb{R} \times \Omega$ endowed with the compound metric $(1-\chi_T)\Big(dx \otimes dx + \mathfrak{h}\Big) + \chi_T \mathsf{g}$, where $\chi_T(x) = \chi(x-T)$ and T is a sufficiently large positive number. The compound metric is well-defined, because the metric g stabilizes to the product metric $dx \otimes dx + \mathfrak{h}$ at infinity, see Definition 2.2. Let Δ^T be the Laplacian induced on $\mathbb{R} \times \Omega$ by the compound metric. Introduce the conjugated Laplacian $\Delta^T_\beta = e^{-\beta x}\Delta^T e^{\beta x}$, where $e^{\pm\beta x}$ stands for the operator of multiplication by the exponent. Let $\overset{\circ}{H}{}^2(\mathbb{R} \times \Omega)$ and $L^2(\mathbb{R} \times \Omega)$ be the spaces introduced in the proof of Proposition 5.3. The Dirichlet Laplacian $\Delta^T_\beta : \overset{\circ}{H}{}^2(\mathbb{R} \times \Omega) \to L^2(\mathbb{R} \times \Omega)$ possesses the properties:

1. Δ^T_β coincides with ${}^0\Delta_\beta$ on $(T+1, \infty) \times \Omega \subset \Pi$.
2. Δ^T_β coincides with $\Delta_\Omega - (\partial_x + \beta)^2$ on $(-\infty, T) \times \Omega$.
3. As a consequence of stabilization of g to $dx \otimes dx + \mathfrak{h}$ at infinity the estimate

   $$\|(\Delta^T_\beta - \Delta_\Omega + (\partial_x + \beta)^2)u\|_{L^2(\mathbb{R}\times\Omega)} \leqslant C(T)\|u\|_{H^2(\mathbb{R}\times\Omega)} \tag{7.1}$$

   is valid, where the constant $C(T)$ is independent of $u \in \overset{\circ}{H}{}^2(\mathbb{R} \times \Omega)$ and $\beta \in \mathcal{K}$, where $\mathcal{K}$ is a compact subset of $\mathbb{C}$. Moreover, $C(T) \to 0$ as $T \to +\infty$, cf. (4.9) and (4.10).

Let us show that for arbitrarily small $\delta > 0$ there exists a sufficiently large $T > 0$, such that the resolvent

$$(\Delta^T_\beta - \mu)^{-1} \in \mathscr{B}(L^2(\mathbb{R} \times \Omega), \overset{\circ}{H}{}^2(\mathbb{R} \times \Omega)) \tag{7.2}$$

is an analytic function of β on the compact subset

$$\mathcal{K}_\delta = \{\beta \in \mathbb{C} : -\delta \geqslant \Re\beta \geqslant \delta - \min_{j:\nu_j>\mu} \sqrt{\nu_j - \mu},\ |\Im\beta| \leqslant \delta\}$$

of the complex plane; i.e. it is analytic in a small open neighborhood of $\mathcal{K}_\delta$. Recall from the proof of Proposition 5.3 that the operator (5.7) yields an isomorphism, if $\beta \in \mathbb{C}$ does not satisfy the equality (5.4) with any $j \in \mathbb{N}$ and any $\xi \in \mathbb{R}$. Thus the operator (5.7) with $\lambda = 0$ and $\mu \in \mathbb{R} \setminus \{\nu_j\}_{j=1}^\infty$ is an analytic Fredholm operator function of $\beta \in \mathcal{K}_\delta$, which is invertible for all $\beta \in \mathcal{K}_\delta$. The analytic Fredholm theory immediately implies that the inverse operator

$$(\Delta_\Omega - (\partial_x + \beta)^2 - \mu)^{-1} : L^2(\mathbb{R} \times \Omega) \to \overset{\circ}{H}{}^2(\mathbb{R} \times \Omega)$$

is uniformly bounded (analytic) function of $\beta \in \mathcal{K}_\delta$. From this together with (7.1) we conclude that for all sufficiently large $T > T(\delta)$ and all $\beta \in \mathcal{K}_\delta$ the operator norm of the composition

$$\Lambda = (\Delta_\Omega - (\partial_x + \beta)^2 - \Delta_\beta^T)(\Delta_\Omega - (\partial_x + \beta)^2 - \mu)^{-1} \in \mathscr{B}(L^2(\mathbb{R} \times \Omega))$$

is less than one. Therefore for all $\beta \in \mathcal{K}_\delta$ we have

$$(\Delta_\beta^T - \mu)^{-1} = (\Delta_\Omega - (\partial_x + \beta)^2 - \mu)^{-1} \sum_{j=0}^\infty \Lambda^j,$$

where the series $\sum_{j=0}^\infty \Lambda^j$ converges in the space $\mathscr{B}(L^2(\mathbb{R}\times\Omega))$. As Δ_β^T depends on $\beta \in \mathbb{C}$ analytically, and the inclusion (7.2) is valid for all $\beta \in \mathcal{K}_\delta$, the resolvent (7.2) is an analytic function of $\beta \in \mathcal{K}_\delta$.

Let Ψ meet the assumptions of theorem. Thanks to (3.1) for any $\beta \in \mathbb{C}$ with $\Re\beta > \gamma$ we have $e^{-\beta \mathsf{s}}\Psi \in \ker(\Delta_\beta - \mu)$. Then for $\varphi_T = \chi_{T+2}$ we obtain

$$(\Delta_\beta^T - \mu)\varphi_T e^{-\beta \mathsf{s}}\Psi = e^{-\beta \mathsf{s}}[\Delta, \varphi_T]\Psi.$$

(Here the right hand side of the equality reads as follows: the cutoff function φ_T is extended from its support in Π to the manifold $\mathcal{M}$ by zero, then the commutator is well-defined, and we extend the right hand side of the equality from its support in Π to $\mathbb{R}\times\Omega$ by zero.) Consequently, for all $\beta \in \mathcal{K}_\delta$, $\Re\beta > \gamma$, the equality

$$\varphi_T e^{-\beta \mathsf{s}}\Psi = (\Delta_\beta^T - \mu)^{-1} e^{-\beta \mathsf{s}}[\Delta, \varphi_T]\Psi \tag{7.3}$$

is valid. The function $[\Delta, \varphi_T]\Psi \in C^\infty(\mathbb{R} \times \Omega)$ is compactly supported, and hence $e^{-\beta \mathsf{s}}[\Delta, \varphi_T]\Psi$ is an analytic function of $\beta \in \mathbb{C}$ with values in $L^2(\mathbb{R}\times\Omega)$. Since the resolvent (7.2) is analytic in $\beta \in \mathcal{K}_\delta$, the right hand side of (7.3) is an analytic function of $\beta \in \mathcal{K}_\delta$ with values in the space $\overset{\circ}{H}{}^2(\mathbb{R} \times \Omega)$. Hence the function $\mathcal{K}_\delta \ni \beta \mapsto \varphi_T e^{-\beta \mathsf{s}}\Psi \in \overset{\circ}{H}{}^2(\mathbb{R} \times \Omega)$ is analytic. As δ is arbitrarily small, this establishes the inclusion $\varphi_T e^{-\beta \mathsf{s}}\Psi \in \overset{\circ}{H}{}^2(\mathbb{R} \times \Omega)$, where $\Re\beta > -\min_{j:\nu_j>\mu} \sqrt{\nu_j - \mu}$. Now we can conclude that Ψ meets the pointwise estimate (3.1) with an independent of x constant C and $\gamma > -\min_{j:\nu_j>\mu} \sqrt{\nu_j - \mu}$.

Indeed, let $\hat{u}(\tau) = \mathcal{F}_{x\mapsto\tau} u(x)$ be the Fourier transform of $u = \varphi_T e^{-\beta \mathsf{s}}\Psi$. Then $\int_{\mathbb{R}} (1+\tau)^4 \|\hat{u}(\tau)\|^2_{L^2(\Omega)}\, d\tau \leqslant c\|u\|_{H^2(\mathbb{R}\times\Omega)}$ e.g. [19]. For $u(x) = \int_{\mathbb{R}} e^{ix\tau}\hat{u}(\tau)\, d\tau$ we

deduce the estimates

$$\|u(x)\|_{L^2(\Omega)} \leqslant \int_{\mathbb{R}} \|e^{ix\tau}\hat{u}(\tau)\|_{L^2(\Omega)}\,d\tau \leqslant \Big(\int_{\mathbb{R}} (1+\tau)^{-2}\,d\tau\Big)^{1/2}$$
$$\times \Big(\int_{\mathbb{R}} (1+\tau)^2 \|\hat{u}(\tau)\|^2_{L^2(\Omega)}\,d\tau\Big)^{1/2} \leqslant C\|u\|_{H^2(\mathbb{R}\times\Omega)},$$

where we used the Cauchy-Schwarz inequality. This completes the proof for the case of the Dirichlet Laplacian.

In order to study the case of the Neumann Laplacian, we consider the continuous operator

$$\begin{aligned}\{\Delta_\Omega - &(1+\lambda)^{-2}(\partial_x+\beta)^2 - \mu, \partial_\eta\} : \\ &H^2(\mathbb{R}\times\Omega) \to L^2(\mathbb{R}\times\Omega) \times H^{1/2}(\mathbb{R}\times\partial\Omega)\end{aligned} \tag{7.4}$$

of the non-homogeneous Neumann problem on $(\mathbb{R}\times\Omega, dx\otimes dx+\mathfrak{h})$. Here ∂_η is the operator of the Neumann boundary conditions, the space $H^\ell(\mathbb{R}\times\Omega)$ is introduced as the completion of the set $C_c^\infty(\mathbb{R}\times\Omega)$ in the norm (5.6), and $H^{1/2}(\mathbb{R}\times\partial\Omega)$ is the space of traces of the functions in $H^1(\mathbb{R}\times\Omega)$. Applying the Fourier transform $\mathcal{F}_{x\mapsto\xi}$ we pass from the operator (7.4) to the operator $\{\Delta_\Omega + (1+\lambda)^{-2}(\beta+i\xi)^2 - \mu, \partial_\eta\}$ of the non-homogeneous Neumann problem on $(\Omega,\mathfrak{h})$. Suppose that μ does not satisfy the equality (5.3) for any $j\in\mathbb{N}$ and $\xi\in\mathbb{R}$ or, equivalently, suppose that for any $\xi\in\mathbb{R}$ the number $\mu-(1+\lambda)^{-2}(\beta+i\xi)^2$ is not an eigenvalue ν_j of the Neumann Laplacian Δ_Ω on $(\Omega,\mathfrak{h})$. Then a known argument, see e.g. [20, Theorem 4.1] or [18, Theorem 5.2.2] or [17, Theorem 2.4.1], shows that the operator (7.4) realizes an isomorphism.

In the same way as before we introduce the compound metric on $\mathbb{R}\times\Omega$ and consider the corresponding operator ∂_ν^T of the Neumann boundary condition on $\mathbb{R}\times\partial\Omega$. The continuous operator

$$\{\Delta_\beta^T - \mu, e^{-\beta x}\,\partial_\nu^T\, e^{\beta x}\} : H^2(\mathbb{R}\times\Omega) \to L^2(\mathbb{R}\times\Omega)\times H^{1/2}(\mathbb{R}\times\partial\Omega) \tag{7.5}$$

possesses the properties:

1. It coincides with $\{{}^0\Delta_\beta - \mu, e^{-\beta\mathsf{s}}\partial_\nu e^{\beta\mathsf{s}}\}$ on $(T+1,\infty)\times\Omega\subset\Pi$.
2. It coincides with $\{\Delta_\Omega - (\partial_x+\beta)^2 - \mu, \partial_\eta\}$ on $(-\infty,T)\times\Omega$.
3. As a consequence of stabilization of g to $dx\otimes dx+\mathfrak{h}$ at infinity the estimate

$$\begin{aligned}\|(\Delta_\beta^T - \Delta_\Omega + (\partial_x+\beta)^2)u\|_{L^2(\mathbb{R}\times\Omega)}& \\ + \|\partial_\eta u - e^{-\beta x}\,\partial_\nu^T\, e^{\beta x} u\|_{H^{1/2}(\mathbb{R}\times\partial\Omega)} &\leqslant C(T)\|u\|_{H^2(\mathbb{R}\times\Omega)}\end{aligned}$$

   is valid, where the constant $C(T)$ is independent of $u\in H^2(\mathbb{R}\times\Omega)$ and $\beta\in\mathcal{K}_\delta$; moreover, $C(T)\to 0$ as $T\to+\infty$.

Similarly to the case of the Dirichlet Laplacian one can show that for any $\delta>0$ there exists $T>T(\delta)$, such that the inverse of the operator (7.5) is an

analytic function of $\beta \in \mathcal{K}_\delta$. Therefore the equality

$$\varphi_T e^{-\beta \mathsf{s}}\Psi = \{\Delta_\beta^T - \mu, e^{-\beta x}\,\partial_\nu^T\, e^{\beta x}\}^{-1} e^{-\beta \mathsf{s}}\{[\Delta,\varphi_T]\Psi, [\partial_\nu,\varphi_T]\Psi\}$$

extends by analyticity from the set $\{\beta \in \mathcal{K}_\delta : \Re\beta > \gamma\}$ to all $\beta \in \mathcal{K}_\delta$. This equality is a substitution for (7.3), it implies the inclusion $\chi e^{-\beta \mathsf{s}}\Psi \in H^2(\mathbb{R}\times\Omega)$, where $\beta > -\min_{j:\nu_j>\mu}\sqrt{\nu_j - \mu}$. $\square$

Theorem 7.2 (Accumulation of eigenvalues) *Let $\{\mu_k\}_{k=1}^\infty$ be a sequence of eigenvalues of the Laplacian Δ on a manifold $(\mathcal{M}, \mathsf{g})$ with an asymptotically cylindrical end, such that $\mu_k \to \mu < \infty$ as $k \to \infty$, and $\mu_k \neq \mu$. Assume that to every μ_k there corresponds an eigenfunction Ψ_k satisfying the estimate* (3.1) *with some $\gamma = \gamma_k < 0$. Then $\gamma_k \to 0$, μ is a threshold ν_j of Δ, and the sequence $\{\mu_k\}_{k=1}^\infty$ accumulates to ν_j only from below.*

PROOF. Here we combine the ideas used in the proof of Theorem 7.1 with the compactness argument due to Perry [27].

Let us first show that $\gamma_k \to 0$ as $k \to \infty$. Assume the contrary. Then there exists $\beta < 0$, such that $\gamma_k < \beta$ for all $k > 0$. By taking a larger negative β we can always achieve $\mu \notin \sigma_{ess}({}^0\Delta_\beta)$, cf. Proposition 5.3 and Fig. 4.

In the case of the Dirichlet Laplacian, and also in the case $\partial\mathcal{M} = \varnothing$, similarly to (7.3) we deduce

$$\begin{aligned}\varphi_T e^{-\beta \mathsf{s}}\Psi_k &= (\Delta_\beta^T - \mu)^{-1}(\Delta_\beta - \mu_k + \mu_k - \mu)\varphi_T e^{-\beta \mathsf{s}}\Psi_k \\ &= (\Delta_\beta^T - \mu)^{-1}\Big(e^{-\beta \mathsf{s}}[\Delta,\varphi_T] + (\mu_k - \mu)\varphi_T e^{-\beta \mathsf{s}}\Big)\Psi_k.\end{aligned}$$

For all sufficiently large T the resolvent (7.2) is bounded by the argument in the proof of Theorem 7.1. Since the metric g stabilizes to the product metric at infinity, the norm $\|f\|$ of a function supported in Π is equivalent to the norm $\|f\|_{L^2(\mathbb{R}\times\Omega)}$, and the norm in $H^2(\mathcal{M})$ is equivalent to the graph norm of the Laplacian, see the proof of Proposition 5.3. Hence

$$\|\varphi_T e^{-\beta \mathsf{s}}\Psi_k\|^2 \leqslant C(\mu_k^2+1)\|\Psi_k\|^2 + C(\mu_k-\mu)^2\|\varphi_T e^{-\beta \mathsf{s}}\Psi_k\|^2, \tag{7.6}$$

where we extended the functions from their supports to $\mathcal{M}$ by zero and used the estimates

$$\|e^{-\beta \mathsf{s}}[\Delta,\varphi_T]\Psi_k\|^2 \leqslant c_1\|\Psi_k\|^2_{H^2(\mathcal{M})} \leqslant c_2(\|\Delta\Psi_k\|^2 + \|\Psi_k\|^2) = c_2(\mu_k^2+1)\|\Psi_k\|^2.$$

In the case of the Neumann Laplacian we use the equality

$$\varphi_T e^{-\beta \mathsf{s}}\Psi_k = \{\Delta_\beta^T - \mu, e^{-\beta x}\,\partial_\nu^T\, e^{\beta x}\}^{-1} e^{-\beta \mathsf{s}}\{[\Delta,\varphi_T]\Psi_k + (\mu_k-\mu)\Psi_k, [\partial_\nu,\varphi_T]\Psi_k\}.$$

This equality together with boundedness of the inverse of (7.5) and the estimates

$$\|e^{-\beta \mathsf{s}}[\Delta,\varphi_T]\Psi_k\|^2+\|e^{-\beta \mathsf{s}}[\partial_\nu,\varphi_T]\Psi_k\|^2_{H^{1/2}(\mathbb{R}\times\partial\Omega)}\leqslant c_1\|\Psi_k\|^2_{H^2(\mathcal{M})}$$
$$\leqslant c_2(\|\Delta\Psi_k\|^2+\|\Psi_k\|^2)=c_2(\mu_k^2+1)\|\Psi_k\|^2$$

justifies the uniform in k inequality (7.6) for the Neumann Laplacian.

Clearly, $1-C(\mu_k-\mu)^2>0$ for all sufficiently large k, and from (7.6) we get

$$\|\varphi_T e^{-\beta \mathsf{s}}\Psi_k\|^2\leqslant C(\mu_k^2+1)\Big(1-C(\mu_k-\mu)^2\Big)^{-1}\|\Psi_k\|^2\leqslant Const\|\Psi_k\|^2. \quad (7.7)$$

For an independent of $\Psi\in L^2(\mathcal{M})$ constant C we have $\|(1-\varphi_T)e^{-\beta \mathsf{s}}\Psi\|\leqslant C\|\Psi\|$. Thus (7.7) leads to the uniform in k estimate $\|e^{-\beta \mathsf{s}}\Psi_k\|\leqslant \mathrm{C}\|\Psi_k\|$. This estimate together with the Cauchy-Schwarz inequality gives

$$\|\Psi_k\|^2\leqslant\|e^{-\beta \mathsf{s}}\Psi_k\|\|e^{\beta \mathsf{s}}\Psi_k\|\leqslant \mathrm{C}\|\Psi_k\|\,\|e^{\beta \mathsf{s}}\Psi_k\|.$$

Finally, for all sufficiently large k we obtain the uniform estimate

$$\|\Psi_k\|^2_{H^2(\mathcal{M})}\leqslant C(\|\Delta\Psi_k\|^2+\|\Psi_k\|^2)=C(\mu_k^2+1)\|\Psi_k\|^2\leqslant Const\|e^{\beta \mathsf{s}}\Psi_k\|^2. \quad (7.8)$$

The Sobolev space $H^2(\mathcal{M})$ is compactly embedded into the weighted space $L^2(\mathcal{M},e^{\beta \mathsf{s}})$. Since compactness of the unit ball implies that the space is finite-dimensional, there can only be a finite number of linearly independent normalized eigenfunctions Ψ_k satisfying (7.8). Since $\mu_k\to\mu\neq\mu_k$ as $k\to\infty$, we come to a contradiction.

We proved that $\gamma_k\to 0$ as $k\to\infty$. It remains to note that for any subsequence of $\{\mu_k\}_{k=1}^\infty$ that does not accumulate to a threshold ν_j from below we can use Theorem 7.1 in order to refine the corresponding subsequence of $\{\gamma_k\}_{k=1}^\infty$ so that the elements γ_k of the subsequence does not tend to zero as $k\to\infty$. This completes the proof.

Let us remark here that an independent proof of this theorem can be obtained by methods announced in [15] and then developed in [13,29] and [14, Appendix]. □

Proof of Theorem 3.1. The assertions are readily apparent from Theorems 6.2, 7.1, 7.2, and Corollary 5.4. □

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